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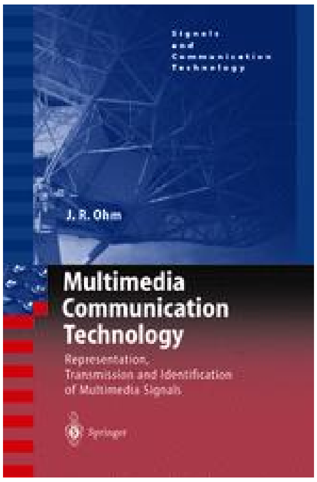
Multimedia Communication Technology

Solutions on End-of-chapter Problems



Problem 1.1

a) per image frame: 720 x 576 + 2 · (360 x 576) Byte = 829,440 Byte
per second: 829,440 Byte · 25 = 20.736 Mbyte
per minute: 20.736 Mbyte · 60 = 1.24416 Gbyte
whole movie: 1.24416 Gbyte · 150 = 186.624 Gbyte

b) 186.624 x (1:40) = 4.6656 GByte

c) 20.736 Mbyte/s = 20.736 · 8 Mbit/s = 165.888 Mbit/s
165.888 Mbit/s · (1:40) = 4.1472 Mbit/s using MPEG-2
Increase by 10%: 4.1472 Mbit/s · 1.1 = 4.56192 Mbit/s

Problem 1.2

a) Rate for video signal: 56 kbit/s – 16 kbit/s = 40 kbit/s
buffer memory size: 40 kbit/s · 200 ms = 8 kbit

b) Mean number of bits per frame is 40 kbit/s : 10 frames/s = 4 kbit/frame (target data rate). Decrease of Q factor by k effects increase of actual data rate by $(1.1)^k$ – if the Q factor is increased, k is less than zero, such that the data rate becomes lower. Effect of the rate control: updated data rate = actual data rate · $(1.1)^k$ ≤target
4.84 kbit · $(1.1)^k \leq$ 4 kbit $\Rightarrow (1.1)^k \leq$ 4/4.84 kbit $\Rightarrow k \cdot \log(1.1) \leq \log(4/4.84)$
$k \leq \log(4/4.84) / \log(1.1) = -2$
Consequently, the Q factor must be increased by two steps.

Problem 2.1

a) $\mathbf{A} = \begin{bmatrix} \mathbf{a} & \mathbf{b} \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}$; $\tilde{\mathbf{A}} = \begin{bmatrix} \tilde{\mathbf{a}} & \tilde{\mathbf{b}} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \tilde{\mathbf{a}}^{\mathrm{T}} \\ \tilde{\mathbf{b}}^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} \tilde{a}_0 & \tilde{a}_1 \\ \tilde{b}_0 & \tilde{b}_1 \end{bmatrix}$; $\tilde{\mathbf{A}}\mathbf{A} = \mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\tilde{\mathbf{a}}^{\mathrm{T}}\mathbf{a} = 1 \quad \Rightarrow \quad 2\tilde{a}_0 = 1 \quad \Rightarrow \quad \tilde{a}_0 = \frac{1}{2}$$

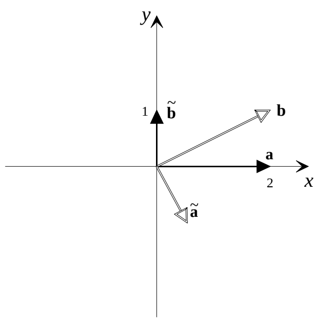
$$\tilde{\mathbf{a}}^{\mathrm{T}}\mathbf{b} = 0 \Rightarrow 2\tilde{a}_0 + \tilde{a}_1 = 0 \Rightarrow \tilde{a}_1 = -1$$

$$\tilde{\mathbf{b}}^{\mathrm{T}}\mathbf{a} = 0 \Rightarrow 2\tilde{b}_0 = 0 \Rightarrow \tilde{b}_0 = 0$$

$$\tilde{\mathbf{b}}^{\mathrm{T}}\mathbf{b} = 1 \Rightarrow 2\tilde{b}_0 + \tilde{b}_1 = 1 \Rightarrow \tilde{b}_1 = 1$$

$$\tilde{\mathbf{A}} = \begin{bmatrix} \frac{1}{2} & -1 \\ 0 & 1 \end{bmatrix}$$

b) see Figure ....



c) $|\mathbf{A}| = 2$; $|\tilde{\mathbf{A}}| = \frac{1}{2} = \frac{1}{|\mathbf{A}|}$

Problem 2.2

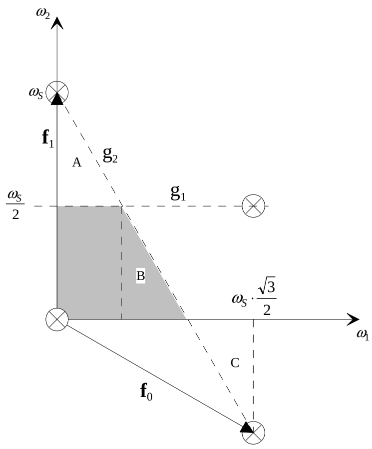
a) With the frequency domain sampling basis $\mathbf{F} = \omega_S \cdot \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 \\ -\frac{1}{2} & 1 \end{bmatrix}$

Interpretation from the following Figure: Boundary of base band (gray) in the first quadrant is limited by lines g_1 and g_2. The centers of basebands and of next periodic spectrum copies

⊗ establish corners of triangles having side lengths $|\mathbf{f}_0|=|\mathbf{f}_1|=|\omega_S|$. Triangles A,B and C are congruent.

The intercepts of the lines are as given in the following table:

	ω_1 axis	ω_2 axis
g_1	parallel	$\omega_S/2$
g_2	$\omega_S \cdot \sqrt{3}/3$	ω_S



The line equations are

$$g_1: \quad \omega_2 = \frac{\omega_S}{2}$$

$$g_2: \quad \frac{\omega_1}{\omega_S \cdot \sqrt{3}/3} + \frac{\omega_2}{\omega_S} = 1 \quad \Leftrightarrow \quad \omega_1 + \frac{\omega_2}{\sqrt{3}} = \frac{\omega_S}{\sqrt{3}}$$

All signal components above any of the two lines must be zero to fulfill the sampling conditions. In the first quadrant, this can be formulated as

$$X(j\omega_1, j\omega_2) = 0 \text{ for } \omega_2 \geq \frac{\omega_S}{2} \text{ and } \omega_1 + \frac{\omega_2}{\sqrt{3}} \geq \frac{\omega_S}{\sqrt{3}}.$$

Generalization into all four quadrants gives (2.62).

b) From Figure above: Gray area in first quadrant (rectangle + triangle B having half size of rectangle)

$$A_{hex} = \frac{\omega_S}{2} \cdot \frac{\sqrt{3}}{6} \cdot \omega_S \cdot 1.5 = \frac{\omega_S^{\,2}}{4} \cdot \frac{\sqrt{3}}{2}$$

In comparison, using rectangular sampling with $\omega_R=\omega_S$ in the first quadrant (from Fig. 2.13c)

$$A_{rect} = \left(\frac{\omega_S}{2}\right)^2 \Rightarrow \frac{A_{hex}}{A_{rect}} = \frac{\sqrt{3}}{2} \approx 0.866$$

c) From Fig. 2.12c resp. length of the basis vector $\mathbf{d}_0$ from (2.54) : $R_{hex} = S \cdot \frac{2}{\sqrt{3}}$.

For rectangular sampling $R_{rect} = S \quad \Rightarrow \frac{R_{hex}}{R_{rect}} = \frac{2}{\sqrt{3}}$.

d) $$\left|\mathbf{D}_{hex}\right| = \frac{2}{\sqrt{3}} \cdot 1 - \frac{1}{\sqrt{3}} \cdot 0 = \frac{2}{\sqrt{3}} = \frac{R_{hex}}{R_{rect}} = \frac{1}{\left(\frac{A_{hex}}{A_{rect}}\right)}$$

Results of c) and d) determine the factor, by which the number of samples is lower in case of hexagonal sampling, as compared to rectangular sampling using identical line spacing. The result from b) is the reciprocal value, as the size of the baseband spectrum behaves reciprocally with the 'sampling area'.

e) Area of the CCD chip 10x7.5 mm^2, $M \cdot N=4 \cdot 10^6$ Pixel.

Rectangular: $R = S \Rightarrow M_{rect} = \frac{10}{7.5}\cdot N_{rect} \Rightarrow \frac{4}{3}\cdot N_{rect}\cdot N_{rect} = 4\cdot 10^6$

$\Rightarrow N_{rect} = \sqrt{3}\cdot 10^3 \approx 1{,}732$ rows

Hexagonal: $R = \frac{2}{\sqrt{3}}\cdot S \Rightarrow M_{hex} = \frac{10}{7.5}\cdot\frac{\sqrt{3}}{2}\cdot N_{hex} \Rightarrow \frac{2}{\sqrt{3}}\cdot N_{hex}\cdot N_{hex} = 4\cdot 10^6$

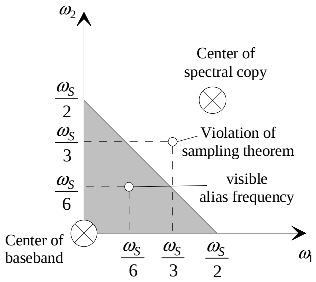
$\Rightarrow N_{hex} = \sqrt{2\cdot\sqrt{3}}\cdot 10^3 \approx 1{,}861$ rows

Sampling distances:
rectangular 7.5 mm / 1.732 = 4.33 μm ; hexagonal 7.5 mm / 1.861 = 4.03 μm.

Problem 2.3

a) Boundaries of the baseband: $|\tilde{\omega}_1| + |\tilde{\omega}_2| = \frac{\omega_S}{2} \Rightarrow |\omega_{2,\max}| + \frac{\omega_S}{3} < \frac{\omega_S}{2} \Rightarrow |\omega_{2,\max}| < \frac{\omega_S}{6}$.

b) According to the result from a), the sampling condition is violated. Alias in the first quadrant results from the spectral copy at $\tilde{\omega}_1 = \omega_S/2$, $\tilde{\omega}_2 = \omega_S/2$. The resulting alias frequency is $\omega_1 = \omega_S/2 - \omega_S/3 = \omega_S/6$, $\omega_2 = \omega_S/2 - \omega_S/3 = \omega_S/6$.



Problem 2.4

a) Sampling distances:

$T = \frac{1}{50\ Hz} = 20\ ms$; $R = \frac{10\ mm}{360} = 27.77\ \mu m$; $S = \frac{7.5\ mm}{288} = 26.04\ \mu m$

Velocities:

$u = k\cdot\frac{R}{T} = 20\cdot\frac{27.77\ \mu m}{20\ ms} = 27.77\frac{mm}{s}$; $v = l\cdot\frac{S}{T} = 10\cdot\frac{26.04\ \mu m}{20\ ms} = 13.02\frac{mm}{s}$

b) Length of a cosine period : 40 samples $\Rightarrow \tilde{\omega}_2 = \frac{2\pi}{40S} = \frac{\omega_S}{40}$.

Limit condition in the first quadrant (positive frequency) $\omega_{3,\max} < \frac{\pi}{T}$.

with $\omega_3 = \omega_1\cdot u + \omega_2\cdot v \Rightarrow \omega_{1,\max}\cdot u + \frac{\omega_S}{40}\cdot v < \frac{\pi}{T}$.

Substitution by normalized frequencies and pixel shifts:

$\omega_1 = \Omega_1\frac{\omega_R}{2\pi} = \frac{\Omega_1}{R}$; $\omega_2 = \frac{\Omega_2}{S}$; $u = k\cdot\frac{R}{T}$; $v = l\cdot\frac{S}{T}$

$$\Rightarrow \frac{\Omega_{1,\max}}{R}\cdot k\cdot\frac{R}{T}+\frac{2\pi}{40\cdot S}\cdot l\cdot\frac{S}{T}<\frac{\pi}{T}$$

$$\Rightarrow \Omega_{1,\max}\cdot 20+\frac{\pi}{2}<\pi \Rightarrow \Omega_{1,\max}<\frac{\pi}{40} \text{ resp. } \omega_{1,\max}<\frac{\omega_R}{80}$$

Problem 2.5

a) Example of a possible sampling matrix : $\mathbf{D}=\begin{bmatrix}2R & R & R\\ 0 & S & 0\\ 0 & 0 & T\end{bmatrix}=\begin{bmatrix}\mathbf{d}_0 & \mathbf{d}_1 & \mathbf{d}_2\end{bmatrix}$.

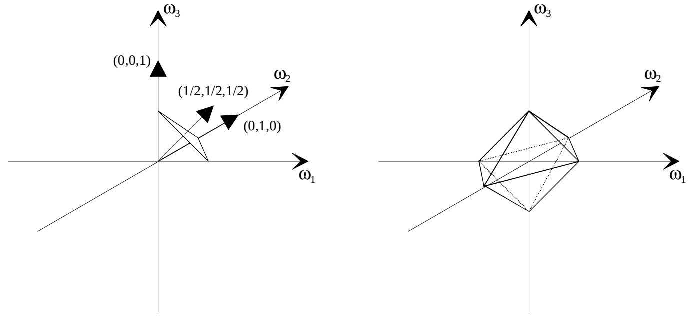
b) Condition $\mathbf{F}^{\mathrm{T}}\mathbf{D}=\mathbf{I}$ resp. $\begin{bmatrix}\mathbf{f}_0\\ \mathbf{f}_1\\ \mathbf{f}_2\end{bmatrix}\cdot\begin{bmatrix}\mathbf{d}_0 & \mathbf{d}_1 & \mathbf{d}_2\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

$$\mathbf{f}_0{}^{\mathrm{T}}\cdot\mathbf{d}_0=1 \Leftrightarrow 2R\cdot f_{0,0}=1 \quad \Rightarrow f_{0,0}=1/(2R)$$
$$\mathbf{f}_0{}^{\mathrm{T}}\cdot\mathbf{d}_1=0 \Leftrightarrow R\cdot f_{0,0}+S\cdot f_{1,0}=0 \quad \Rightarrow f_{1,0}=-1/(2S)$$
$$\mathbf{f}_0{}^{\mathrm{T}}\cdot\mathbf{d}_2=0 \Leftrightarrow R\cdot f_{0,0}+T\cdot f_{2,0}=0 \quad \Rightarrow f_{2,0}=-1/(2T)$$
$$\mathbf{f}_1{}^{\mathrm{T}}\cdot\mathbf{d}_0=0 \Leftrightarrow 2R\cdot f_{0,1}=0 \quad \Rightarrow f_{0,1}=0$$
$$\mathbf{f}_1{}^{\mathrm{T}}\cdot\mathbf{d}_1=1 \Leftrightarrow R\cdot f_{0,1}+S\cdot f_{1,1}=1 \quad \Rightarrow f_{1,1}=1/S$$
$$\mathbf{f}_1{}^{\mathrm{T}}\cdot\mathbf{d}_2=0 \Leftrightarrow R\cdot f_{0,1}+T\cdot f_{2,1}=0 \quad \Rightarrow f_{2,1}=0$$
$$\mathbf{f}_2{}^{\mathrm{T}}\cdot\mathbf{d}_0=0 \Leftrightarrow 2R\cdot f_{0,2}=0 \quad \Rightarrow f_{0,2}=0$$
$$\mathbf{f}_2{}^{\mathrm{T}}\cdot\mathbf{d}_1=0 \Leftrightarrow R\cdot f_{0,2}+S\cdot f_{1,2}=0 \quad \Rightarrow f_{1,2}=0$$
$$\mathbf{f}_2{}^{\mathrm{T}}\cdot\mathbf{d}_2=1 \Leftrightarrow R\cdot f_{0,2}+T\cdot f_{2,2}=1 \quad \Rightarrow f_{2,2}=1/T$$

$$\Rightarrow \mathbf{F}=\left[\mathbf{D}^{-1}\right]^{\mathrm{T}}=\begin{bmatrix}\frac{1}{2R} & 0 & 0\\ -\frac{1}{2S} & \frac{1}{S} & 0\\ -\frac{1}{2T} & 0 & \frac{1}{T}\end{bmatrix}$$

c) For simplicity, the sketch is made for normalized sampling distances R=S=T=1, and for a dual basis system **F**' with all-positive vectors in the 3D frequency space:

$$\mathbf{F}'=\begin{bmatrix}1/2 & 0 & 0\\ 1/2 & 1 & 0\\ 1/2 & 0 & 1\end{bmatrix}=\left[\begin{bmatrix}2 & -1 & -1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}^{-1}\right]^{\mathrm{T}}$$



The Figure above shows (left) the boundary plane of the baseband in the first octant of the 3D frequency space, intersecting the basis vectors at their respective centers. The volume of the baseband over all eight octants is an octaeder (right). Consequently, intersecting planes at ω_1=0, ω_2=0 and ω_3=0 result for pairs of frequency components ω_2/ω_3, ω_1/ω_3 and ω_1/ω_2 in a baseband shape of 2D quincunx sampling, respectively.

Problem 2.6

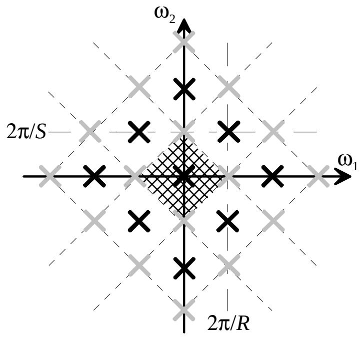
a) Quincunx grid: R=S; R'=S'=S/2. Hexagonal grid: $R = S\cdot\sqrt{3}; R' = S\cdot\sqrt{3}/2; S' = S/2$. This corresponds to the grid shown in Fig. 2.19, which differs from Fig. 2.12c in terms of grid positions (rotated by 30^0). Possible basis matrices are

$$\mathbf{D}_{hex} = S\begin{bmatrix} \frac{\sqrt{3}}{2} & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \quad \text{or} \quad \mathbf{D}_{hex} = S\begin{bmatrix} \sqrt{3} & \frac{\sqrt{3}}{2} \\ 0 & \frac{1}{2} \end{bmatrix}.$$

b) $\delta_{\bullet\circ}(\mathbf{r}) = \delta_{R,S}(r,s) + \delta_{R,S}(r-R', s-S')$

c) $\mathscr{F}\{\delta_{\bullet\circ}(\mathbf{r})\} = \frac{4\pi^2}{RS}\delta_{\omega_R,\omega_S}(\omega_1,\omega_2)\cdot\left(1+e^{-j(\omega_1 R'+\omega_2 S')}\right) \quad \omega_R = \frac{2\pi}{R}; \quad \omega_S = \frac{2\pi}{S}$

For any frequency position $k\omega_R R'+l\omega_S S'=(2m+1)\pi$ (k,l,m) integer (shown light gray), the members of the Dirac impulse grid are cancelled out. The quincunx grid as shown in the Figure remains.



d) Condition in frequency domain: If the Fourier transform of $\delta_{\bullet\circ}(\mathbf{r})$ is not real-valued. Condition in spatial domain: If any linear combination of basis vectors does not point to a valid sampling position, e.g. for R=S, R'=S'=S/4.

e) See Figure above.

f) Observed signal: $x_{alias}(r,s) = A + B$

Problem 3.1

a) $E\left\{[x(\mathbf{n}) \pm x(\mathbf{n}+\mathbf{k})]^2\right\} \geq 0 \quad \Rightarrow \quad E\left\{[(x(\mathbf{n})-\mu_x) \pm (x(\mathbf{n}+\mathbf{k})-\mu_x)]^2\right\} \geq 0$

$$E\left\{[x(\mathbf{n})-\mu_x]^2\right\} \pm 2\cdot E\{[x(\mathbf{n})-\mu_x]\cdot[x(\mathbf{n}+\mathbf{k})-\mu_x]\} + E\left\{[x(\mathbf{n}+\mathbf{k})-\mu_x]^2\right\} \geq 0$$

$$\Rightarrow 2\left[\sigma_x^2 \pm r'_{xx}(\mathbf{k})\right] \geq 0 \quad \Rightarrow \sigma_x^2 \geq |r'_{xx}(\mathbf{k})|$$

b) $r_{xx}(\mathbf{k}) = E\{\, x(\mathbf{n})\cdot x(\mathbf{n}+\mathbf{k})\} = E\{\, x(\mathbf{n}+\mathbf{k}')\cdot x(\mathbf{n}+\mathbf{k}+\mathbf{k}')\}$ for all $\mathbf{k}'$

$$\mathbf{k}'=-\mathbf{k}: \quad r_{xx}(\mathbf{k})=E\left\{x(\mathbf{n}-\mathbf{k})\cdot x(\mathbf{n})\right\}=r_{xx}(-\mathbf{k})$$

c) $$r'_{xy}(\mathbf{k})=E\left\{\left[x(\mathbf{n})-\mu_x\right]\cdot\left[y(\mathbf{n}+\mathbf{k})-\mu_y\right]\right\}$$

$$=E\left\{x(\mathbf{n})\cdot y(\mathbf{n}+\mathbf{k})\right\}-E\left\{x(\mathbf{n})\right\}\cdot\mu_y-E\left\{y(\mathbf{n}+\mathbf{k})\right\}\cdot\mu_x+\mu_x\cdot\mu_y$$

$$=r_{xy}(\mathbf{k})-\mu_x\mu_y \quad \text{with } E\left\{x(\mathbf{n})\right\}=\mu_x \text{ and } E\left\{y(\mathbf{n}+\mathbf{k})\right\}=\mu_y$$

d) $$\frac{d}{dy}\left[E\left\{(x(\mathbf{n})-y)^2\right\}\right]=\frac{d}{dy}\left[E\left\{x^2(\mathbf{n})\right\}-2y\cdot E\left\{x(\mathbf{n})\right\}+y^2\right]=-2\cdot E\left\{x(\mathbf{n})\right\}+2\cdot y=0$$

$$\Rightarrow \quad y=E\{x(\mathbf{n})\}$$

e) $$E\left\{\mathbf{x}\cdot\mathbf{x}^{\mathrm{T}}\right\}=E\left\{\begin{bmatrix}x(n)\\x(n+1)\\x(n+2)\end{bmatrix}\cdot\begin{bmatrix}x(n) & x(n+1) & x(n+2)\end{bmatrix}\right\}$$

$$=\begin{bmatrix}E\{x(n)\cdot x(n)\} & E\{x(n)\cdot x(n+1)\} & E\{x(n)\cdot x(n+2)\}\\E\{x(n+1)\cdot x(n)\} & E\{x(n+1)\cdot x(n+1)\} & E\{x(n+1)\cdot x(n+2)\}\\E\{x(n+2)\cdot x(n)\} & E\{x(n+2)\cdot x(n+1)\} & E\{x(n+2)\cdot x(n+2)\}\end{bmatrix}$$

$$=\begin{bmatrix}r_{xx}(0) & r_{xx}(1) & r_{xx}(2)\\r_{xx}(1) & r_{xx}(0) & r_{xx}(1)\\r_{xx}(2) & r_{xx}(1) & r_{xx}(0)\end{bmatrix}$$

Problem 3.2

$$p(x)=a\cdot e^{-|bx|^{\gamma}} \quad ; \quad a=\frac{b\gamma}{2\Gamma\left(\frac{1}{\gamma}\right)} \quad ; \quad b=\frac{1}{\sigma_x}\sqrt{\frac{\Gamma\left(\frac{3}{\gamma}\right)}{\Gamma\left(\frac{1}{\gamma}\right)}} \quad ; \quad \Gamma(x)=\int_0^\infty e^{-t}y^{x-1}dy$$

a) $$b=\frac{1}{\sigma_x}\cdot\sqrt{\frac{\left(\frac{\sqrt{\pi}}{2}\right)}{\sqrt{\pi}}}=\frac{1}{\sigma_x}\cdot\sqrt{\frac{1}{2}}=\frac{1}{\sqrt{2\sigma_x^{\,2}}}\ ; \ a=\frac{\frac{1}{\sqrt{2\sigma_x^{\,2}}}\cdot 2}{2\cdot\sqrt{\pi}}=\frac{1}{\sqrt{2\pi\sigma_x^{\,2}}}$$

$$\Rightarrow p(x)=\frac{1}{\sqrt{2\pi\sigma_x^{\,2}}}\cdot e^{-\frac{x^2}{2\sigma_x^{\,2}}}$$

b) $$b=\frac{1}{\sigma_x}\cdot\sqrt{\frac{2}{1}}=\frac{\sqrt{2}}{\sigma_x} \quad ; \quad a=\frac{\left(\frac{\sqrt{2}}{\sigma_x}\right)}{2}=\frac{1}{\sqrt{2}\cdot\sigma_x} \quad \Rightarrow p(x)=\frac{1}{\sqrt{2}\cdot\sigma_x}\cdot e^{-\frac{\sqrt{2}\cdot|x|}{\sigma_x}}$$

c) $$\lim_{x\to\infty}\Gamma\left(\frac{3}{x}\right)=\Gamma(0)\ ;\ \lim_{x\to\infty}\Gamma\left(\frac{1}{x}\right)=\Gamma(0)\ ;\ \text{but } \Gamma(0)=\lim_{x\to 0}\frac{\Gamma(1)}{x}=\lim_{x\to 0}\frac{1}{x}=\infty$$

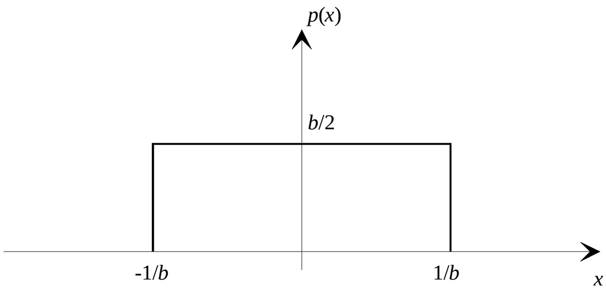
$$b=\frac{1}{\sigma_x}\sqrt{\frac{\Gamma(3/\infty)}{\Gamma(1/\infty)}}=?$$

Assumption : b is a positive and real-valued constant. Different cases are:

$$|x|<\frac{1}{b} \quad \Rightarrow |bx|<1 \quad \Rightarrow |bx|^{\infty}=0 \quad \Rightarrow e^{-|bx|^{\infty}}=1$$

$$|x|>\frac{1}{b} \quad \Rightarrow |bx|>1 \quad \Rightarrow |bx|^{\infty}=\infty \quad \Rightarrow e^{-|bx|^{\infty}}=0$$

This is a uniform distribution, $p(x)=a=b/2$ for $-1/b<x<1/b$.



The constants a and b are: $\sigma_x^{\ 2}=\frac{b}{2}\int_{-1/b}^{1/b}x^2dx=\frac{1}{3b^2}\Rightarrow b=\frac{1}{\sqrt{3}\cdot\sigma_x}\ ;\ a=\frac{b}{2}=\frac{1}{\sqrt{12}\cdot\sigma_x}$

Problem 3.3

a) $r'_{xx}(k)=\sigma_x^{\ 2}\cdot\rho^{|k|}\Rightarrow r'_{xx}(0)=\sigma_x^{\ 2}\ ;$
$r'_{xx}(1)=\sigma_x^{\ 2}\cdot\rho\ ;\ r'_{xx}(2)=\sigma_x^{\ 2}\cdot\rho^2$ $\quad \mathbf{R}'_{xx}=\sigma_x^{\ 2}\begin{bmatrix}1 & \rho & \rho^2\\ \rho & 1 & \rho\\ \rho^2 & \rho & 1\end{bmatrix}$

b) $\mathbf{R}'_{xx}=\sigma_x^{\ 2}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\Rightarrow[\mathbf{R}'_{xx}]^{-1}=\frac{1}{\sigma_x^{\ 2}}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ $\quad$ Det$|\mathbf{R}'_{xx}|=[\sigma_x^2]^3$

$$p_3(\mathbf{x})=\frac{1}{\sqrt{(2\pi)^3\cdot(\sigma_x^{\ 2})^3}}\cdot e^{-\frac{1}{2\sigma_x^{\ 2}}\cdot\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}^{\mathrm{T}}\cdot\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\cdot\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}}=\frac{1}{\sqrt{(2\pi)^3\cdot(\sigma_x^{\ 2})^3}}\cdot e^{-\frac{1}{2\sigma_x^{\ 2}}\cdot\left[x_1^{\ 2}+x_2^{\ 2}+x_3^{\ 2}\right]}$$

$$=\frac{1}{\left[\sqrt{(2\pi)\cdot(\sigma_x^{\ 2})}\right]^3}\cdot e^{-\frac{x_1^{\ 2}}{2\sigma_x^{\ 2}}}\cdot e^{-\frac{x_2^{\ 2}}{2\sigma_x^{\ 2}}}\cdot e^{-\frac{x_3^{\ 2}}{2\sigma_x^{\ 2}}}=p_G(x_1)\cdot p_G(x_2)\cdot p_G(x_3)$$

Problem 3.4

a) For the example of a 2D Fourier transform:

$X(j\Omega_1, j\Omega_2) = \sum_{m=-\infty}^{\infty}\sum_{n-\infty}^{\infty} x(m,n)\cdot e^{-jm\Omega_1}\cdot e^{-jn\Omega_2}$ is real-valued for $e^{-j\cdots}=\pm 1$, which occurs independent of the m,n for $\Omega_1=k\cdot\pi$, $\Omega_2=l\cdot\pi$. Mapping into discrete frequencies u and v of the DFT with $\Omega_1 = u\cdot\frac{2\pi}{M}$; $\Omega_2 = v\cdot\frac{2\pi}{N}$ gives as candidates $\Omega_i=0$ and $\Omega_i=\pi$, with $0\le u<M$, $0\le v<N$. Substitution gives

$$0 = u\cdot\frac{2\pi}{M} \quad\Rightarrow u=0 \quad ; \quad 0 = v\cdot\frac{2\pi}{N} \quad\Rightarrow v=0$$

$$\pi = u\cdot\frac{2\pi}{M} \quad\Rightarrow u=\frac{M}{2} \quad ; \quad \pi = v\cdot\frac{2\pi}{N} \quad\Rightarrow v=\frac{N}{2}$$

The lower two conditions are only fulfilled for even-valued M,N, as u,v must be integer. For odd DFT analysis lengths, no DFT spectral line is found at exactly half sampling frequency.

i. M and N even: Real values for $u=0,v=0$; $u=M/2,v=0$; $u=0,v=N/2$; $u=M/2,v=N/2$.

ii. M even, N odd: Real values for $u=0,v=0$; $u=M/2,v=0$.

iii. M odd, N even : Real values for $u=0,v=0$; $u=0,v=N/2$.

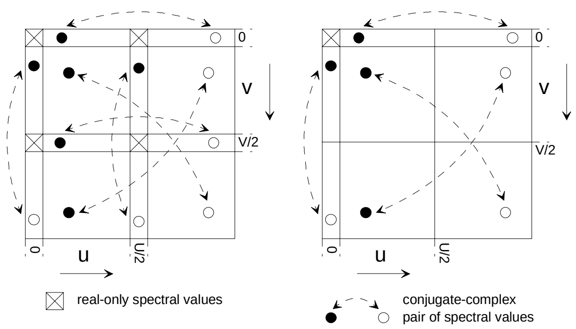
iv. M and N odd : Real values for $u=0,v=0$.

b) $$\hat{X}^*(u,v) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} x(m,n)\cdot W_M{}^{mu}\cdot W_N{}^{nv} \quad mit \quad W_M = e^{j\frac{2\pi}{M}} \quad und \quad W_N = e^{j\frac{2\pi}{N}}$$

$$\hat{X}(M-u, N-v) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} x(m,n)\cdot W_M{}^{-m(M-u)}\cdot W_N{}^{-n(N-v)}$$

$$= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} x(m,n)\cdot W_M{}^{mu}\cdot W_M{}^{-mM}\cdot W_N{}^{nv}\cdot W_N{}^{-nN}$$

With $W_M{}^{-mM} = e^{-j\frac{2\pi}{M}\cdot mM} = e^{-j2\pi m} = 1$ and $W_N{}^{-nN} = e^{-j\frac{2\pi}{N}\cdot nN} = e^{-j2\pi n} = 1$ equality of the two expressions is given. The positions of conjugate-complex pairs in different areas of the spectrum and the positions of real-only values are shown for cases i. (a) and iv. (b) in the following Figure.



a) b)

Problem 3.5

a) For statistical independency, $P(j_1,j_2)=P(j_1)\cdot P(j_2)$, $P(j_1|j_2)=P(j_1)$, $P(j_2|j_1)=P(j_2)$. Hence,

$$H(j_2 \mid j_1) = -\sum_{j_1}\sum_{j_2} P(j_1)\cdot P(j_2)\cdot \log_2 P(j_2)$$
$$= -\underbrace{\sum_{j_1} P(j_1)}_{=1}\cdot \sum_{j_2} P(j_2)\cdot \log_2 P(j_2) = H(j_2),$$

similarly for $H(j_1|j_2)$. This gives

$$I(j_1;j_2) = H(j_2) - H(j_2 \mid j_1) = H(j_1) - H(j_1 \mid j_2) = 0\,,$$

which can also be shown directly:

$$I(j_1;j_2) = \sum_{j_1}\sum_{j_2} P(j_1)\cdot P(j_2)\cdot \log_2 \frac{P(j_1)\cdot P(j_2)}{P(j_1)\cdot P(j_2)} = 0$$

b) For this case, $P(j_1,j_2)=P(j_1)\cdot\delta(j_1-j_2)=P(j_2)\cdot\delta(j_1-j_2)$, $P(j_1|j_2)=P(j_2|j_1)=\delta(j_1-j_2)$ (discrete delta impulse: =0 for $j_1\neq j_2$, =1 else). Hence, all entries $j_1\neq j_2$ can be disregarded:

$$H(j_1 \mid j_2) = \sum_{j_1(j_1=j_2)} P(j_1)\cdot \log_2(1) = 0 \quad \Rightarrow I(j_1;j_2) = H(j_1) - H(j_1 \mid j_2) = H(j_1)$$

which can also be shown directly:

$$I(j_1;j_2) = \sum_{j_1(j_1=j_2)} P(j_1)\cdot \log_2 \frac{P(j_1)}{P(j_1)\cdot P(j_1)} = \sum_{j_1(j_1=j_2)} P(j_1)\cdot \log_2 \frac{1}{P(j_1)} = H(j_1)$$

Problem 3.6

a) $\left|\mathbf{R}'_{xy}\right| = \sigma_x{}^2\sigma_y{}^2 - r'_{xy}{}^2$ $\qquad \mathbf{R}'_{xy}{}^{-1} = \frac{1}{\left|\mathbf{R}'_{xy}\right|}\cdot\begin{bmatrix} \sigma_y{}^2 & -r'_{xy} \\ -r'_{xy} & \sigma_x{}^2 \end{bmatrix}$

$$p(x,y) = \frac{1}{\sqrt{(2\pi)^2\left|\mathbf{R}'_{xy}\right|}}\cdot e^{-\frac{1}{2}\cdot\frac{\sigma_y{}^2\cdot(x-\mu_x)^2+\sigma_x{}^2\cdot(y-\mu_y)^2-2r'_{xy}\cdot(x-\mu_x)\cdot(y-\mu_y)}{\sigma_x{}^2\cdot\sigma_y{}^2-r'_{xy}{}^2}}$$

$r'_{xy}=0$: (in the sequel, assumption of zero-mean signals, $\mu_x=\mu_y=0$)

$$p(x,y) = \frac{1}{\sqrt{(2\pi)^2\cdot\sigma_x{}^2\cdot\sigma_y{}^2}}\cdot e^{-\frac{1}{2}\cdot\frac{\sigma_y{}^2\cdot x^2+\sigma_x{}^2\cdot y^2}{\sigma_x{}^2\cdot\sigma_y{}^2}} = \frac{1}{\sqrt{2\pi\cdot\sigma_x{}^2}}\cdot\frac{1}{\sqrt{2\pi\cdot\sigma_x{}^2}}\cdot e^{-\frac{1}{2}\cdot\frac{x^2}{\sigma_x{}^2}}\cdot e^{-\frac{1}{2}\cdot\frac{y^2}{\sigma_y{}^2}}$$
$$= p(x)\cdot p(y)$$

$x=y$: As joint values with $x\neq y$ cannot occur, the following condition holds:

$$\int_{-\infty}^{y-\varepsilon} p(x,y)dx = \int_{y+\varepsilon}^{\infty} p(x,y)dx = 0 \text{ for all } \varepsilon > 0 \quad\text{and}\quad \int_{-\infty}^{\infty} p(x,y)dy = p(x)$$

Using the "sifting property" of $\delta(\cdot)$, these conditions are fulfilled by the following function using the Dirac impulse: $p(x,y)=p(x)\cdot\delta(y-x)$. This also guarantees that the volume below the joint PDF is one:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y)dydx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x)\cdot\delta(y-x)dydx = \int_{-\infty}^{\infty} p(x)\cdot \underbrace{\int_{-\infty}^{\infty}\delta(y-x)dy}_{=1}dx = \int_{-\infty}^{\infty} p(x)dx = 1$$

Formally (not strictly analytically) it can also be shown that the joint PDF possesses a singularity (Dirac impulse!) for $x=y$, which however scales with a Gaussian parametrized over x. With $\sigma_x^2=\sigma_y^2=r'_{xy}$, which holds true for $x=y$:

$$p(x,y)\big|_{y=x} = \frac{1}{\sqrt{(2\pi)^2\cdot\left(\sigma_x{}^2\cdot\sigma_x{}^2-\sigma_x{}^2\cdot\sigma_x{}^2\right)}}\cdot e^{-\frac{1}{2}\cdot\frac{\sigma_x{}^2\cdot x^2+\sigma_x{}^2\cdot x^2-2\sigma_x{}^2\cdot x^2}{\sigma_x{}^2\cdot\sigma_x{}^2-\sigma_x{}^2\cdot\sigma_x{}^2}}$$

$$= \frac{1}{\sqrt{(2\pi)^2\cdot\sigma_x{}^2\cdot\left(\sigma_x{}^2-\sigma_x{}^2\right)}}\cdot e^{-\frac{1}{2}\cdot\frac{2\left(\sigma_x{}^2-\sigma_x{}^2\right)\cdot x^2}{\left(\sigma_x{}^2-\sigma_x{}^2\right)\cdot\left(\sigma_x{}^2+\sigma_x{}^2\right)}} = \underbrace{\frac{1}{\sqrt{2\pi\cdot\left(\sigma_x{}^2-\sigma_x{}^2\right)}}}_{\to\infty}\cdot\underbrace{\frac{1}{\sqrt{2\pi\sigma_x{}^2}}\cdot e^{-\frac{1}{2}\cdot\frac{x^2}{\sigma_x{}^2}}}_{p_G(x)}$$

b) $$p(y\,|\,x) = \frac{p(x,y)}{p(x)} = \frac{1}{\sqrt{(2\pi)^2\cdot\left(\sigma_x{}^2\cdot\sigma_y{}^2-r'_{xy}{}^2\right)}}\cdot e^{-\frac{1}{2}\cdot\frac{\sigma_y{}^2\cdot x^2+\sigma_x{}^2\cdot y^2-2r'_{xy}\cdot xy}{\sigma_x{}^2\cdot\sigma_y{}^2-r'_{xy}{}^2}}\cdot\sqrt{2\pi\sigma_x{}^2}\cdot e^{\frac{1}{2}\cdot\frac{x^2}{\sigma_x{}^2}}$$

Expressed by normalized covariance coefficients $\rho'_{xy}=r'_{xy}/(\sigma_x\sigma_y)$:

$$p(y\,|\,x) = \frac{\sqrt{2\pi\sigma_x{}^2}}{\sqrt{(2\pi)^2\cdot\sigma_x{}^2\cdot\sigma_y{}^2\cdot\left(1-\rho'_{xy}{}^2\right)}}\cdot e^{-\frac{1}{2}\left(\frac{\sigma_y{}^2\cdot x^2+\sigma_x{}^2\cdot y^2-2\sigma_x\sigma_y\rho'_{xy}\cdot xy}{\sigma_x{}^2\cdot\sigma_y{}^2\cdot\left(1-\rho'_{xy}{}^2\right)}-\frac{x^2}{\sigma_x{}^2}\right)}$$

$$= \frac{1}{\sqrt{2\pi\cdot\sigma_y{}^2\cdot\left(1-\rho'_{xy}{}^2\right)}}\cdot e^{-\frac{1}{2}\cdot\underbrace{\frac{\sigma_y{}^2\cdot x^2+\sigma_x{}^2\cdot y^2-2\sigma_x\sigma_y\rho'_{xy}\cdot xy-x^2\cdot\sigma_y{}^2\cdot\left(1-\rho'_{xy}{}^2\right)}{\sigma_x{}^2\cdot\sigma_y{}^2\cdot\left(1-\rho'_{xy}{}^2\right)}}_{=\frac{\sigma_x{}^2\cdot y^2-2\sigma_x\sigma_y\rho'_{xy}\cdot xy+x^2\cdot\sigma_y{}^2\cdot\rho'_{xy}{}^2}{\sigma_x{}^2\cdot\sigma_y{}^2\left(1-\rho'_{xy}{}^2\right)}}}$$

$$= \frac{1}{\sqrt{2\pi\cdot\sigma_y{}^2\cdot\left(1-\rho'_{xy}{}^2\right)}}\cdot e^{-\frac{1}{2}\cdot\frac{\left(\sigma_x\cdot y-\sigma_y\rho'_{xy}\cdot x\right)^2}{\sigma_x{}^2\cdot\sigma_y{}^2\cdot\left(1-\rho'_{xy}{}^2\right)}} = \frac{1}{\sqrt{2\pi\cdot\sigma_y{}^2\cdot\left(1-\rho'_{xy}{}^2\right)}}\cdot e^{-\frac{1}{2}\cdot\frac{\left(y-\frac{\sigma_y}{\sigma_x}\rho'_{xy}\cdot x\right)^2}{\sigma_y{}^2\cdot\left(1-\rho'_{xy}{}^2\right)}}$$

For uncorrelated signals, $\rho'_{xy}=0$:

$$p(y\,|\,x) = \frac{1}{\sqrt{2\pi\cdot\sigma_y{}^2}}\cdot e^{-\frac{1}{2}\cdot\frac{y^2}{\sigma_y{}^2}} = p(y)$$

Problem 4.1

a) $$\begin{bmatrix}1 & \rho\\ \rho & 1\end{bmatrix}\cdot\begin{bmatrix}\phi_{u,0}\\ \phi_{u,1}\end{bmatrix} = \lambda_u\cdot\begin{bmatrix}\phi_{u,0}\\ \phi_{u,1}\end{bmatrix} \quad ; \quad u=0,1$$

u=0 : $\phi_{0,0} + \rho \cdot \phi_{0,1} = (1+\rho)\cdot \phi_{0,0}$; $\rho \cdot \phi_{0,0} + \phi_{0,1} = (1+\rho)\cdot \phi_{0,1} \Rightarrow \phi_{0,0} = \phi_{0,1}$

u=1 : $\phi_{1,0} + \rho \cdot \phi_{1,1} = (1-\rho)\cdot \phi_{1,0}$; $\rho \cdot \phi_{1,0} + \phi_{1,1} = (1-\rho)\cdot \phi_{1,1} \Rightarrow \phi_{1,0} = -\phi_{1,1}$

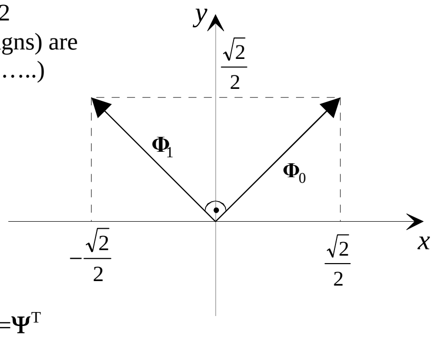
Orthonormality:

$$\phi_{0,0}{}^2 + \phi_{0,1}{}^2 = 1 \quad ; \quad \phi_{1,0}{}^2 + \phi_{1,1}{}^2 = 1 \quad \Rightarrow \quad \phi_{0,0}{}^2 = \phi_{0,1}{}^2 = \phi_{1,0}{}^2 = \phi_{1,1}{}^2 = \frac{1}{2}$$

$$\phi_{0,0} = \phi_{0,1} = \pm\frac{\sqrt{2}}{2} \quad ; \quad \phi_{1,0} = -\phi_{1,1} = \pm\frac{\sqrt{2}}{2}$$

b) 4 possible solutions (combinations of ± signs) are (one of which is shown here………………..)

$$\mathbf{\Phi}_0 = \pm\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} ; \ \mathbf{\Phi}_1 = \pm\begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}$$



c) Basis system is orthonormal and real: $\mathbf{\Psi}^{-1}=\mathbf{\Psi}^{T}$

$$\mathbf{\Psi} = [\ \mathbf{\Phi}_0 \quad \mathbf{\Phi}_1\ ] = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \mathbf{\Psi}^{-1}$$

d) $|\mathbf{R}|=1-\rho^2$; $\lambda_1\cdot\lambda_2=(1+\rho)\cdot(1-\rho)=1-\rho^2=|\mathbf{\Lambda}|$ in the eigenvalue system

$$\mathbf{R}\cdot\mathbf{\Psi} = \mathbf{\Psi}\cdot\mathbf{\Lambda} \quad ; \quad \mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

Problem 4.2

Proof of autocorrelation properties:

$$x_{\mathrm{AR}}(n+1) = \rho\cdot x_{\mathrm{AR}}(n) + z(n+1)$$

$$x_{\mathrm{AR}}(n+2) = \rho\cdot x_{\mathrm{AR}}(n+1) + z(n+2) = \rho^2\cdot x_{\mathrm{AR}}(n) + \rho\cdot z(n+1) + z(n+2)$$

$$x_{\mathrm{AR}}(n+k) = \rho\cdot x_{\mathrm{AR}}(n+k-1) + z(n+k) = \rho^k\cdot x_{\mathrm{AR}}(n) + \sum_{l=1}^{k}\rho^{k-l}\cdot z(n+l)$$

$$r_{xx}(k) = E\left\{x_{\mathrm{AR}}(n)x_{\mathrm{AR}}(n+k)\right\}$$

$$= \rho^k \underbrace{E\left\{x_{\mathrm{AR}}(n)x_{\mathrm{AR}}(n)\right\}}_{=\sigma_x{}^2} + \underbrace{E\left\{x_{\mathrm{AR}}(n)\sum_{l=1}^{k}\rho^{k-l}z(n+l)\right\}}_{=0}$$

$$r_{xx}(-k) = r_{xx}(k) \Rightarrow r_{xx}(k) = \sigma_x{}^2\rho^{|k|}$$

Proof of variance properties:

$$\sigma_z{}^2 = E\left\{z^2(n)\right\} = E\left\{\left(x_{\mathrm{AR}}(n) - \rho\cdot x_{\mathrm{AR}}(n-1)\right)^2\right\}$$

$$= \underbrace{E\left\{\left(x_{\mathrm{AR}}(n)\right)^2\right\}}_{=\sigma_x{}^2} - 2\rho\underbrace{E\left\{x_{\mathrm{AR}}(n)\cdot x_{\mathrm{AR}}(n-1)\right\}}_{=\rho\cdot\sigma_x{}^2} + \rho^2\underbrace{E\left\{\left(x_{\mathrm{AR}}(n-1)\right)^2\right\}}_{=\sigma_x{}^2}$$

$$= \sigma_x{}^2\left(1-\rho^2\right)$$

Proof of spectral properties:

$$B(j\Omega) = \frac{1}{1-\rho\cdot e^{-j\Omega}}$$

$$S_{xx}(\Omega) = \sigma_z^{\ 2}\cdot\left|B(j\Omega)\right|^2 = \frac{\sigma_z^{\ 2}}{\left(1-\rho\cdot e^{-j\Omega}\right)\left(1-\rho\cdot e^{j\Omega}\right)}$$

$$= \frac{\sigma_z^{\ 2}}{1-\rho\cdot\left(e^{-j\Omega}+e^{j\Omega}\right)+\rho^2} = \frac{\sigma_z^{\ 2}}{1-2\rho\cdot\cos\Omega+\rho^2}$$

alternatively by Fourier transform of ACF:

$$S_{xx}(\Omega) = \sum_{k=-\infty}^{\infty} r_{xx}(k)e^{-jk\Omega} = \sigma_x^{\ 2}\sum_{k=-\infty}^{\infty}\rho^{|k|}e^{-jk\Omega} = \sigma_x^{\ 2}\left[\sum_{k=0}^{\infty}\left(\rho e^{-j\Omega}\right)^k + \sum_{k=1}^{\infty}\left(\rho e^{j\Omega}\right)^k\right].$$

With $\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}$ when $|a|<1$:

$$\Rightarrow S_{xx}(\Omega) = \sigma_x^{\ 2}\left[\frac{1}{1-\rho e^{-j\Omega}}+\frac{1}{1-\rho e^{j\Omega}}-1\right] = \frac{\sigma_x^{\ 2}\left(1-\rho^2\right)}{1-\rho\cdot\left(e^{-j\Omega}+e^{j\Omega}\right)+\rho^2} = \frac{\sigma_z^{\ 2}}{1-2\rho\cdot\cos\Omega+\rho^2}$$

Problem 4.3

$$\sigma_x^{\ 2}\cdot\begin{bmatrix}\rho\\ \rho^2\end{bmatrix} = \sigma_x^{\ 2}\cdot\begin{bmatrix}1 & \rho\\ \rho & 1\end{bmatrix}\cdot\begin{bmatrix}a(1)\\ a(2)\end{bmatrix}$$

$$a(1)+\rho\cdot a(2) = \rho \qquad \Rightarrow a(1) = \rho\cdot\left[1-a(2)\right]$$

$$\rho\cdot a(1)+a(2) = \rho^2 \Rightarrow \rho^2\cdot\left[1-a(2)\right]+a(2) = \rho^2 \Rightarrow a(2)\cdot\left[1-\rho^2\right] = 0$$

For $\rho\neq 1$: $a(1)=\rho$; $a(2)=0$. For $\rho=1$: $a(1)+a(2)=1$

Problem 4.4

$$\sigma_e^{\ 2} = E\left\{e^2(n)\right\} = E\left\{\left[x(m,n)-\hat{x}(m,n)\right]^2\right\};\ \hat{x}(m,n) = 0,5\cdot x(m-1,n)+0,5\cdot x(m,n-1)$$

$$\sigma_e^{\ 2} = E\left\{x^2(m,n)\right\}-2\cdot 0,5\cdot E\left\{x(m,n)\cdot x(m-1,n)\right\}-2\cdot 0,5\cdot E\left\{x(m,n)\cdot x(m,n-1)\right\}$$

$$+0,5\cdot 0,5\cdot E\left\{x^2(m-1,n)\right\}+0,5\cdot 0,5\cdot E\left\{x^2(m,n-1)\right\}+2\cdot 0,25\cdot E\left\{x(m-1,n)\cdot x(m,n-1)\right\}$$

$$= \sigma_x^{\ 2}-\rho_h\cdot\sigma_x^{\ 2}-\rho_v\cdot\sigma_x^{\ 2}+0,25\cdot\sigma_x^{\ 2}+0,25\cdot\sigma_x^{\ 2}+0,5\cdot\rho_h\cdot\rho_v\cdot\sigma_x^{\ 2}$$

$$= \sigma_x^{\ 2}\cdot\left[1,5-\rho_h-\rho_v+0,5\cdot\rho_h\cdot\rho_v\right]$$

For $\rho_h=\rho_v=0,95$: $\sigma_e^2=0,05125\cdot\sigma_x^2$

Separable Predictor : Prediction error = variance of innovation, according to (4.61)

$\sigma_e^{\ 2} = \sigma_z^{\ 2} = \sigma_x^{\ 2}\cdot\left[1-\rho_h^{\ 2}\right]\cdot\left[1-\rho_v^{\ 2}\right]$. For $\rho_h=\rho_v=0,95$: $\sigma_e^2=0,009506\cdot\sigma_x$

Problem 4.5

a) Result as in Problem 4.4:

$\sigma_e^{\ 2} = \sigma_z^{\ 2} = \sigma_x^{\ 2}\cdot\left[1-\rho_h^{\ 2}\right]\cdot\left[1-\rho_v^{\ 2}\right]$. For $\rho_h=\rho_v=0,95$: $\sigma_e^2=0,009506\cdot\sigma_x^2$.

b) As a result of translation shift: $x(m,n,o-1)=x(m+k,n+l,o)$

$$\sigma_e{}^2 = E\left\{\left[x(m,n,o) - x(m,n,o-1)\right]^2\right\} = E\left\{\left[x(m,n,o) - x(m+k,n+l,o)\right]^2\right\}$$
$$= E\left\{x^2(m,n,o)\right\} - 2E\left\{x(m,n,o)\cdot x(m+k,n+l,o)\right\} + E\left\{x^2(m+k,n+l,o)\right\}$$
$$= 2\sigma_x{}^2 \cdot \left[1 - \rho_h{}^{|k|}\cdot\rho_v{}^{|l|}\right]$$

For $\rho_h=\rho_v=0{,}95$: $\sigma_e^2=0{,}8025\cdot\sigma_x^2$

c) By knowing actual shift for prediction: $x(m\text{-}k,n\text{-}l,o\text{-}1)=x(m,n,o)$

$$\sigma_e{}^2 = E\left\{\left[x(m,n,o) - x(m-k,n-l,o-1)\right]^2\right\} = E\left\{\left[x(m,n,o) - x(m,n,o)\right]^2\right\} = 0$$

Problem 4.6

a) $\mathbf{T}^{\text{Haar(4)}} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}$; $\mathbf{T}^{\text{Walsh(4)}} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$

b) $\mathbf{C} = \left[\mathbf{T}_v \cdot \mathbf{X}\right]\cdot\mathbf{T}_h{}^{\mathrm{T}}$

i) Haar basis :

Vertical transform step $\mathbf{C}_v = \mathbf{T}_v \cdot \mathbf{X}$

$$\mathbf{C}_v = \frac{1}{2}\cdot\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}\cdot\begin{bmatrix} 18 & 4 & 2 & 4 \\ 18 & 4 & 2 & 4 \\ 2 & 4 & 2 & 4 \\ 2 & 4 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 20 & 8 & 4 & 8 \\ 16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Horizontal transform step $\mathbf{C} = \mathbf{C}_v \cdot \mathbf{T}_h{}^{\mathrm{T}}$

$$\mathbf{C} = \frac{1}{2}\cdot\begin{bmatrix} 20 & 8 & 4 & 8 \\ 16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\cdot\begin{bmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{bmatrix} = \begin{bmatrix} 20 & 8 & 6\sqrt{2} & -2\sqrt{2} \\ 8 & 8 & 8\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

ii) Walsh-Basis :

Vertical transform step $\mathbf{C}_v = \mathbf{T}_v \cdot \mathbf{X}$

$$\frac{1}{2}\cdot\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}\cdot\begin{bmatrix} 18 & 4 & 2 & 4 \\ 18 & 4 & 2 & 4 \\ 2 & 4 & 2 & 4 \\ 2 & 4 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 20 & 8 & 4 & 8 \\ 16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Horizontal transform step $\mathbf{C} = \mathbf{C}_v \cdot \mathbf{T}_h{}^{\mathrm{T}}$

$$\mathbf{C} = \frac{1}{2}\cdot\begin{bmatrix} 20 & 8 & 4 & 8 \\ 16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\cdot\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 20 & 8 & 8 & 4 \\ 8 & 8 & 8 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

c) Haar transform better suitable for this case, as more coefficients are zero, which typically requires less rate for encoding.

Problem 4.7

a) $$\mathbf{T}^{\cos} = \sqrt{\frac{2}{3}} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \cos\frac{\pi}{6} & \cos\frac{\pi}{2} & -\cos\frac{\pi}{6} \\ \cos\frac{\pi}{3} & \cos\pi & \cos\frac{\pi}{3} \end{bmatrix} = \sqrt{\frac{2}{3}} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{3}}{2} & 0 & -\frac{\sqrt{3}}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \mathbf{t}_0^{\mathrm{T}} \\ \mathbf{t}_1^{\mathrm{T}} \\ \mathbf{t}_2^{\mathrm{T}} \end{bmatrix}$$

$$\mathbf{t}_0^{\mathrm{T}} \cdot \mathbf{t}_1 = \left(\sqrt{\frac{2}{3}}\right)^2 \cdot \left[\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}\right] = 0$$

$$\mathbf{t}_0^{\mathrm{T}} \cdot \mathbf{t}_2 = \left(\sqrt{\frac{2}{3}}\right)^2 \cdot \left[\frac{\sqrt{2}}{2}\cdot\frac{1}{2} - \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2}\right] = 0$$

$$\mathbf{t}_1^{\mathrm{T}} \cdot \mathbf{t}_2 = \left(\sqrt{\frac{2}{3}}\right)^2 \cdot \left[\frac{1}{2}\cdot\frac{\sqrt{3}}{2} - \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right] = 0$$

$$\mathbf{t}_0^{\mathrm{T}} \cdot \mathbf{t}_0 = \left(\sqrt{\frac{2}{3}}\right)^2 \cdot \left[\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2\right] = \frac{2}{3}\cdot\frac{3}{2} = 1$$

$$\mathbf{t}_1^{\mathrm{T}} \cdot \mathbf{t}_1 = \left(\sqrt{\frac{2}{3}}\right)^2 \cdot \left[\left(\frac{\sqrt{3}}{2}\right)^2 + 0 + \left(\frac{\sqrt{3}}{2}\right)^2\right] = \frac{2}{3}\cdot\frac{3}{2} = 1$$

$$\mathbf{t}_2^{\mathrm{T}} \cdot \mathbf{t}_2 = \left(\sqrt{\frac{2}{3}}\right)^2 \cdot \left[\left(\frac{1}{2}\right)^2 + 1 + \left(\frac{1}{2}\right)^2\right] = \frac{2}{3}\cdot\frac{3}{2} = 1$$

b) $$\mathbf{R}_{xx} = \sigma_x^{\,2} \cdot \begin{bmatrix} 1 & \rho & \rho^2 \\ \rho & 1 & \rho \\ \rho^2 & \rho & 1 \end{bmatrix} \quad ; \quad \mathbf{R}_{cc} = [\mathbf{T}\cdot\mathbf{R}_{xx}]\cdot\mathbf{T}^{\mathrm{T}}$$

$$\mathbf{R}'_{cc} = \mathbf{T}\cdot\mathbf{R}_{xx} = \sigma_x^{\,2} \cdot \sqrt{\frac{2}{3}} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{3}}{2} & 0 & -\frac{\sqrt{3}}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 1 & \rho & \rho^2 \\ \rho & 1 & \rho \\ \rho^2 & \rho & 1 \end{bmatrix}$$

$$=\sigma_x{}^2\cdot\sqrt{\frac{2}{3}}\begin{bmatrix}\frac{\sqrt{2}}{2}\cdot\left[1+\rho+\rho^2\right] & \frac{\sqrt{2}}{2}\cdot[1+2\rho] & \frac{\sqrt{2}}{2}\cdot\left[1+\rho+\rho^2\right]\\ \frac{\sqrt{3}}{2}\cdot\left[1-\rho^2\right] & 0 & \frac{\sqrt{3}}{2}\cdot\left[\rho^2-1\right]\\ \frac{1}{2}\cdot\left[1+\rho^2\right]-\rho & \rho-1 & \frac{1}{2}\cdot\left[1+\rho^2\right]-\rho\end{bmatrix}$$

$$\mathbf{R}_{cc}=\mathbf{R}'_{cc}\cdot\mathbf{T}^{\mathrm{T}}=\sigma_x{}^2\cdot\frac{2}{3}\cdot\begin{bmatrix}\frac{\sqrt{2}}{2}\cdot\left[1+\rho+\rho^2\right] & \frac{\sqrt{2}}{2}\cdot[1+2\rho] & \frac{\sqrt{2}}{2}\cdot\left[1+\rho+\rho^2\right]\\ \frac{\sqrt{3}}{2}\cdot\left[1-\rho^2\right] & 0 & \frac{\sqrt{3}}{2}\cdot\left[\rho^2-1\right]\\ \frac{1}{2}\cdot\left[1+\rho^2\right]-\rho & \rho-1 & \frac{1}{2}\cdot\left[1+\rho^2\right]-\rho\end{bmatrix}\cdot\begin{bmatrix}\frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{2} & \frac{1}{2}\\ \frac{\sqrt{2}}{2} & 0 & -1\\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}$$

$$=\sigma_x{}^2\cdot\frac{2}{3}\cdot\begin{bmatrix}\frac{3}{2}+2\rho+\rho^2 & 0 & \frac{\sqrt{2}}{2}\cdot\left[\rho^2-\rho\right]\\ 0 & \frac{3}{2}\cdot\left[1-\rho^2\right] & 0\\ \frac{\sqrt{2}}{2}\cdot\left[\rho^2-\rho\right] & 0 & \frac{3}{2}-2\rho+\frac{\rho^2}{2}\end{bmatrix}$$

c) ρ=0,9 : $\mathbf{R}_{cc,1}=\sigma_x{}^2\cdot\begin{bmatrix}2,8683 & 0 & -0,033\\ 0 & 0,0975 & 0\\ -0,033 & 0 & 0,0342\end{bmatrix}$

ρ=0,5 : $\mathbf{R}_{cc,2}=\sigma_x{}^2\cdot\begin{bmatrix}1,8333 & 0 & -0,177\\ 0 & 0,75 & 0\\ -0,177 & 0 & 0,41671\end{bmatrix}$

In the second case, higher correlation is observed between c_0 and c_2; less concentration of energy in low-frequency coefficients.

d) tr($\boldsymbol{R}_{xx}$) = tr($\mathbf{R}_{cc,1}$) = tr($\mathbf{R}_{cc,2}$) = $3\cdot\sigma_x^2$. As the transformation is orthonormal, the total variance is unchanged.

Problem 4.8

a) $$\mathcal{F}(\mathbf{t}_0)=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\cdot e^{-j\Omega}=e^{-j\frac{\Omega}{2}}\cdot\left[\frac{\sqrt{2}}{2}\cdot e^{j\frac{\Omega}{2}}+\frac{\sqrt{2}}{2}\cdot e^{-j\frac{\Omega}{2}}\right]=e^{-j\frac{\Omega}{2}}\cdot\sqrt{2}\cdot\cos\frac{\Omega}{2}$$

$$\mathcal{F}(\mathbf{t}_1)=\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\cdot e^{-j\Omega}=e^{-j\frac{\Omega}{2}}\cdot\left[\frac{\sqrt{2}}{2}\cdot e^{j\frac{\Omega}{2}}-\frac{\sqrt{2}}{2}\cdot e^{-j\frac{\Omega}{2}}\right]=e^{-j\frac{\Omega}{2}}\cdot\sqrt{2}\cdot j\cdot\sin\frac{\Omega}{2}$$

$$=e^{j\frac{\pi-\Omega}{2}}\cdot\sqrt{2}\cdot\sin\frac{\Omega}{2}$$

b) $$\left(\frac{\sqrt{2}}{2}\right)^2+\left(\pm\frac{\sqrt{2}}{2}\right)^2=1\quad;\quad\left(\frac{\sqrt{2}}{2}\right)^2+\frac{\sqrt{2}}{2}\cdot\left(-\frac{\sqrt{2}}{2}\right)=0$$

c) $$\sqrt{2}\cdot\cos\left(\frac{\Omega}{2}\right)=\sqrt{2}\cdot\sin\left(\frac{\pi}{2}+\frac{\Omega}{2}\right)=\sqrt{2}\cdot\sin\left(\frac{\pi}{2}-\frac{\Omega}{2}\right)$$

$$\left[\text{with}\quad \cos\alpha=\sin\left(\alpha+\pi/2\right);\sin\alpha=\sin(\pi-\alpha)\right]$$

d) $$\left[\sqrt{2}\cdot\cos\frac{\Omega}{2}\right]^2+\left[\sqrt{2}\cdot\sin\frac{\Omega}{2}\right]^2=2\cdot\left[\cos^2\frac{\Omega}{2}+\sin^2\frac{\Omega}{2}\right]=2$$

Problem 4.9

a) Arguments of the windowing function:

$$w(0)=\sin\frac{\pi}{8}=A\quad;\quad w(1)=\sin\frac{3\pi}{8}=B\quad;\quad w(2)=\sin\frac{5\pi}{8}=B\quad;\quad w(3)=\sin\frac{7\pi}{8}=A$$

Arguments of the cosine function:

$$h'_0(0)=\cos\left(-\frac{\pi}{8}\right)=B\ ;\ h'_0(1)=\cos\left(\frac{\pi}{8}\right)=B$$

$$h'_0(2)=\cos\left(\frac{3\pi}{8}\right)=A\ ;\ h'_0(3)=\cos\left(\frac{5\pi}{8}\right)=-A$$

$$h'_1(0)=\cos\left(-\frac{3\pi}{8}\right)=A\ ;\ h'_1(1)=\cos\left(\frac{3\pi}{8}\right)=A$$

$$h'_1(2)=\cos\left(\frac{9\pi}{8}\right)=-B\ ;\ h'_1(3)=\cos\left(\frac{15\pi}{8}\right)=B$$

Basis vectors:

$$\mathbf{t}_0=\left[A\cdot B\quad B\cdot B\quad B\cdot A\quad -A\cdot A\right]^{\mathrm{T}}\quad;\quad \mathbf{t}_1=\left[A\cdot A\quad B\cdot A\quad -B\cdot B\quad A\cdot B\right]^{\mathrm{T}}$$

b) $$\mathbf{t}_0{}^{\mathrm{T}}\cdot\mathbf{t}_1=\mathbf{t}_1{}^{\mathrm{T}}\cdot\mathbf{t}_0=A^3\cdot B+B^3\cdot A-B^3\cdot A-A^3\cdot B=0$$

$$\mathbf{t}_0{}^{\mathrm{T}}\cdot\mathbf{t}_0=A^2\cdot B^2+B^4+B^2\cdot A^2+A^4=\left[A^2+B^2\right]^2$$

$$\mathbf{t}_1{}^{\mathrm{T}}\cdot\mathbf{t}_1=A^4+B^2\cdot A^2+B^4+A^2\cdot B^2=\left[A^2+B^2\right]^2$$

$$\left[A^2+B^2\right]^2=\left[\sin^2\frac{\pi}{8}+\cos^2\frac{\pi}{8}\right]^2=1$$

c) $$\mathcal{F}(\mathbf{t}_0)=AB+B^2\cdot e^{-j\Omega}+AB\cdot e^{-j2\Omega}-A^2\cdot e^{-j3\Omega}=AB\cdot\left[1+e^{-j2\Omega}\right]+e^{-j\Omega}-A^2\cdot\left[e^{-j\Omega}+e^{-j3\Omega}\right]$$

$$=AB\cdot e^{-j\Omega}\cdot\left[e^{j\Omega}+e^{-j\Omega}\right]+e^{-j\Omega}-A^2\cdot e^{-j2\Omega}\cdot\left[e^{j\Omega}+e^{-j\Omega}\right]$$

$$=2AB\cdot e^{-j\Omega}\cdot\cos\Omega+e^{-j\Omega}-2A^2\cdot e^{-j2\Omega}\cdot\cos\Omega=e^{-j\Omega}\cdot\left[1+2\cos\Omega\cdot\left(AB-A^2e^{-j\Omega}\right)\right]$$

$$\mathcal{F}(\mathbf{t}_1)=A^2+AB\cdot e^{-j\Omega}-B^2\cdot e^{-j2\Omega}+AB\cdot e^{-j3\Omega}=1-B^2\cdot\left[1+e^{-j2\Omega}\right]+AB\cdot\left[e^{-j\Omega}+e^{-j3\Omega}\right]$$

$$=1-B^2\cdot e^{-j\Omega}\cdot\left[e^{j\Omega}+e^{-j\Omega}\right]+AB\cdot e^{-j2\Omega}\cdot\left[e^{j\Omega}+e^{-j\Omega}\right]$$

$$=1-2B^2\cdot e^{-j\Omega}\cdot\cos\Omega+2AB\cdot e^{-j2\Omega}\cdot\cos\Omega=1-2e^{-j\Omega}\cdot\cos\Omega\cdot\left(AB\cdot e^{-j\Omega}-B^2\right)$$

d) The basis functions do not express impulse responses of linear-phase filters, as the transfer function is not of the form $T(\Omega)e^{-j\cdots}$. Fast algorithms can be realized, as same factors (A^2,B^2,AB) are used multiple times. By appropriate realization, at most three multiplications/sample must be applied to compute the 1D transform.

Problem 4.10

a) Relationships of filter impulse responses: $h_1(m) = (-1)^{1-m} \cdot h_0(1-m)$

Filter with 6 coefficients, lowpass:

$h_0(-2)$	$h_0(-1)$	$h_0(0)$	$h_0(1)$	$h_0(2)$	$h_0(3)$
A	B	C	C	B	A

Filter with 6 coefficients, highpass:

$h_1(-2)=-h_0(3)$	$h_1(-1)=h_0(2)$	$h_1(0)=-h_0(1)$	$h_1(1)=h_0(0)$	$h_1(2)=-h_0(-1)$	$h_1(3)=h_0(-2)$
-A	B	-C	C	-B	A

$$\sum_{m=-2}^{3} h_0(m) \cdot h_1(m) = -A^2 + B^2 - C^2 + C^2 - B^2 + A^2 = 0$$

$$\sum_{m=-2}^{3} h_u{}^2(m) = 2 \cdot \left(A^2 + B^2 + C^2\right) > 0 \quad ; \quad u = 0{,}1$$

Filter with 5 coefficients, lowpass:

$h_0(-2)$	$h_0(-1)$	$h_0(0)$	$h_0(1)$	$h_0(2)$	$h_0(3)$
A	B	C	B	A	0

Filter with 5 coefficients, highpass:

$h_1(-2)=-h_0(3)$	$h_1(-1)=h_0(2)$	$h_1(0)=-h_0(1)$	$h_1(1)=h_0(0)$	$h_1(2)=-h_0(-1)$	$h_1(3)=h_0(-2)$
0	A	-B	C	-B	A

$$\sum_{m=-2}^{3} h_0(m) \cdot h_1(m) = AB - BC + BC - AB = 0$$

$$\sum_{m=-2}^{3} h_u{}^2(m) = 2 \cdot \left(A^2 + B^2\right) + C^2 > 0 \quad ; \quad u = 0{,}1$$

b)

Filter with 6 coefficients:

z-transfer functions of the equivalent polyphase filter components (without subsampling):

$$H'_{0,A}(z) = H_0(z) = A \cdot z^2 + B \cdot z + C + C \cdot z^{-1} + B \cdot z^{-2} + A \cdot z^{-3}$$

$$H'_{1,A}(z) = H_1(z) = -A \cdot z^2 + B \cdot z - C + C \cdot z^{-1} - B \cdot z^{-2} + A \cdot z^{-3}$$

$$H'_{0,B}(z) = z^{-1} \cdot H_0(z) = A \cdot z + B + C \cdot z^{-1} + C \cdot z^{-2} + B \cdot z^{-3} + A \cdot z^{-4}$$

$$H'_{1,A}(z) = z^{-1} \cdot H_1(z) = -A \cdot z + B - C \cdot z^{-1} + C \cdot z^{-2} - B \cdot z^{-3} + A \cdot z^{-4}$$

Subsampling eliminates the odd-indexed filter coefficients:

$H_{0,A}(z) = A \cdot z + C + B \cdot z^{-1}$

$H_{1,A}(z) = -A \cdot z - C - B \cdot z^{-1}$

$H_{0,B}(z) = B + C \cdot z^{-1} + A \cdot z^{-2}$

$H_{1,A}(z) = B + C \cdot z^{-1} + A \cdot z^{-2}$

Due to $H_{1,A}(z)$=-$H_{0,A}(z)$ and $H_{1,B}(z)$=$H_{0,B}(z)$, the A and B branches each require a filter with 2 delay taps and 3 multiplications. As this is performed in parallel for two samples, only 3 multiplications/sample are necessary effectively (instead of 12, when lowpass and highpass filters are applied directly before the subsampling steps).

Filter with 5 coefficients:
z-transfer functions of the equivalent polyphase filter components (without subsampling):

$H'_{0,A}(z) = H_0(z) = A \cdot z^2 + B \cdot z + C + B \cdot z^{-1} + A \cdot z^{-2}$

$H'_{1,A}(z) = H_1(z) = A \cdot z^1 - B + C \cdot z^{-1} - B \cdot z^{-2} + A \cdot z^{-3}$

$H'_{0,B}(z) = z^{-1} \cdot H_0(z) = A \cdot z + B + C \cdot z^{-1} + B \cdot z^{-2} + A \cdot z^{-3}$

$H'_{1,A}(z) = z^{-1} \cdot H_1(z) = A - B \cdot z^{-1} + C \cdot z^{-2} - B \cdot z^{-3} + A \cdot z^{-4}$

After subsampling:

$H_{0,A}(z) = A \cdot z + C + A \cdot z^{-1} = C + A \cdot \left(z + z^{-1} \right)$

$H_{1,A}(z) = -B - B \cdot z^{-1} = -B \cdot \left(1 + z^{-1} \right)$

$H_{0,B}(z) = B + B \cdot z^{-1} = B \cdot \left(1 + z^{-1} \right)$

$H_{1,A}(z) = A + C \cdot z^{-1} + A \cdot z^{-2} = C \cdot z^{-1} + A \cdot \left(1 + z^{-2} \right)$

Here, again 3 multiplications/sample are necessary (instead of 10 for direct computation). In the A and B branches, each one filter with one and two delay taps must be implemented.

Problem 4.11

a) $\frac{1}{8}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 0 \end{bmatrix} ** \frac{1}{8}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \frac{1}{64}\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 8 & 2 & 0 \\ 1 & 8 & 20 & 8 & 1 \\ 0 & 2 & 8 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$

b) c=1 for (4.248), c=2 for the result (5x5 matrix) of a). The filter of the subsequent pyramid level would have an equivalent effect as a large filter convolving the result of a) by itself: This results in a 9x9 filter matrix, where the coefficients ≠0 establish a diamond-shaped neighborhood system $\mathcal{N}_4^{(1)}$:

$$\frac{1}{64}\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 8 & 2 & 0 \\ 1 & 8 & 20 & 8 & 1 \\ 0 & 2 & 8 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} ** \frac{1}{64}\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 8 & 2 & 0 \\ 1 & 8 & 20 & 8 & 1 \\ 0 & 2 & 8 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= \frac{1}{4096}\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 16 & 4 & 0 & 0 & 0 \\ 0 & 0 & 6 & 48 & 112 & 48 & 6 & 0 & 0 \\ 0 & 4 & 48 & 216 & 400 & 216 & 48 & 4 & 0 \\ 1 & 16 & 112 & 400 & 676 & 400 & 112 & 16 & 1 \\ 0 & 4 & 48 & 216 & 400 & 216 & 48 & 4 & 0 \\ 0 & 0 & 6 & 48 & 112 & 48 & 6 & 0 & 0 \\ 0 & 0 & 0 & 4 & 16 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

c) Computation of the distances related to the center of the filter matrix is performed here row by row, starting at the top-left position:

Filter (4.248) :

$$\sigma^2 = \frac{1}{8}\cdot\left[\ 1\cdot 1^2 + 1\cdot 1^2 + 4\cdot 0^2 + 1\cdot 1^2 + 1\cdot 1^2\right] = \frac{1}{2}.$$

Result of a) :

$$\sigma^2 = \frac{1}{64}\cdot\left[1\cdot2^2+2\cdot\left(\sqrt{2}\right)^2+8\cdot1^2+2\cdot\left(\sqrt{2}\right)^2+1\cdot2^2+8\cdot1^2+20\cdot0^2+8\cdot1^2+1\cdot2^2+2\cdot\left(\sqrt{2}\right)^2+8\cdot1^2+2\cdot\left(\sqrt{2}\right)^2+1\cdot2^2\right]$$

$$= \frac{1}{64}\cdot\left[4\cdot 2^2 + 8\cdot\left(\sqrt{2}\right)^2 + 32\cdot 1^2\right] = \frac{1}{64}\cdot\left[16+16+32\right] = 1.$$

The "variance" of the filter kernel, i.e. the extension of the Gaussian shaped impulse response, is doubled. The transfer function behaves reciprocally, which means that the bandwidth of the signal is reduced by a factor of two with each level of the pyramid.

Problem 4.12

a) $$S_{xx}(\Omega=\pi) = \frac{\sigma_x^{\,2}(1-\rho^2)}{1+2\rho+\rho^2} = \frac{\sigma_x^{\,2}(1-\rho)}{1+\rho} = \frac{\sigma_x^{\,2}}{9} \Rightarrow 1+\rho = 9(1-\rho) \Rightarrow \rho = 0{,}8$$

$$\sigma_z^{\,2} = \sigma_x^{\,2}(1-\rho^2) = 0{,}36\sigma_x^{\,2}$$

b) Prediction of $x(n)$ is performed from $x(n\text{-}2)$. Hence,

$$r_{xx}(2) = \sigma_x^{\,2}\rho^2 \Rightarrow a_{opt} = \frac{r_{xx}(2)}{\sigma_x^{\,2}} = \rho^2$$

c) $$e_g(n') = x(n) - \rho^2 x(n-2), \quad n' = \operatorname{int}\left(n/2\right)$$

$$\sigma_{e_g}^{\,2} = E\left\{\left(x(n)-\rho^2 x(n-2)\right)^2\right\} = \underbrace{E\left\{x^2(n)\right\}}_{\sigma_x^2} - 2\rho^2\underbrace{E\left\{x(n)x(n-2)\right\}}_{\sigma_x^2\rho^2} + \rho^4\underbrace{E\left\{x^2(n-2)\right\}}_{\sigma_x^2}$$

$$= \sigma_x^{\,2}(1-\rho^4) \quad \Rightarrow G = \frac{1}{1-\rho^4}$$

d) $e_g(n') = x(n) - \rho^2 x(n-2); \quad e_o(n') = x(n-1) - \rho^2 x(n-3)$

$$E\left\{e_g(n')e_o(n')\right\} = E\left\{\left[x(n)-\rho^2 x(n-2)\right]\left[x(n-1)-\rho^2 x(n-3)\right]\right\}$$
$$= \underbrace{E\left\{x(n)x(n-1)\right\}}_{\sigma_x^2\rho} - \rho^2 \underbrace{E\left\{x(n-1)x(n-2)\right\}}_{\sigma_x^2\rho}$$
$$-\rho^2 \underbrace{E\left\{x(n)x(n-3)\right\}}_{\sigma_x^2\rho^3} + \rho^4 \underbrace{E\left\{x(n-2)x(n-3)\right\}}_{\sigma_x^2\rho}$$
$$= \sigma_x^2\left(\rho - \rho^3\right)$$

Consequently, the prediction errors within the two polyphase components are *not* uncorrelated.

Problem 4.13

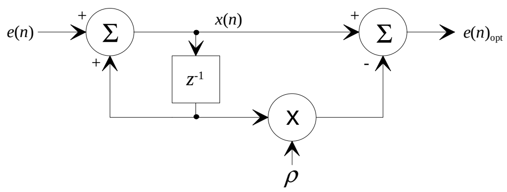
a) $$\sigma_x^2 = \frac{\sigma_z^2}{\left(1-\rho^2\right)} = \frac{7}{\left(1-\frac{9}{16}\right)} = 16$$

b) $$\sigma_e^2 = E\left\{\left(x(n)-x(n-1)\right)^2\right\}$$
$$= E\left\{x^2(n)\right\} - 2E\left\{x(n)\cdot x(n-1)\right\} + E\left\{x^2(n-1)\right\} = 2\sigma_x^2\cdot(1-\rho) = 8$$
$$G_{real} = \frac{\sigma_x^2}{\sigma_e^2} = \frac{16}{8}$$

c) $$G_{opt} = \frac{\sigma_x^2}{\sigma_z^2} = \frac{16}{7} \quad ; \quad \frac{G_{real}}{G_{opt}} = \frac{8}{7}$$

d) $$S_{ee}(\Omega) = S_{xx}(\Omega)\cdot\left|A(j\Omega)\right|^2 = \frac{\sigma_z^2}{1-2\rho\cos\Omega+\rho^2}\cdot\left(1-e^{j\Omega}\right)\cdot\left(1-e^{-j\Omega}\right)$$
$$= \frac{2\sigma_z^2\cdot\left(1-\cos\Omega\right)}{1-2\rho\cos\Omega+\rho^2} = \frac{14\cdot\left(1-\cos\Omega\right)}{\frac{25}{16}-\frac{3}{2}\cos\Omega}$$

e) $$B(z) = \frac{1-\rho\cdot z^{-1}}{1-z^{-1}}$$



Problem 4.14

a) $$\mathbf{T}^{-1} = \frac{1}{\det\mathbf{T}}\begin{bmatrix}-1 & -1/2\\ -1 & 1/2\end{bmatrix} = \begin{bmatrix}1 & 1/2\\ 1 & -1/2\end{bmatrix}$$

b) Basis vectors have different norms (not unity). Equivalent orthonormal transform is $\mathbf{S} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

c) $E\left\{c_0^{\,2}\right\} = \frac{1}{4} E\left\{\left(x(n)+x(n+1)\right)^2\right\} = \frac{1}{4}\left[E\left\{x^2(n)\right\} + 2E\left\{x(n)x(n+1)\right\} + E\left\{x^2(n+1)\right\}\right] = \frac{\sigma_x^{\,2}(1+\rho)}{2}$

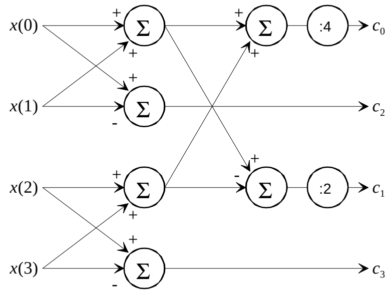
$E\left\{c_1^{\,2}\right\} = E\left\{\left(x(n)-x(n+1)\right)^2\right\} = \left[E\left\{x^2(n)\right\} - 2E\left\{x(n)x(n+1)\right\} + E\left\{x^2(n+1)\right\}\right] = 2\sigma_x^{\,2}(1-\rho)$

d) $\mathbf{y}-\mathbf{x} = \mathbf{T}^{-1}\mathbf{q} = \begin{bmatrix} 1 & 1/2 \\ 1 & -1/2 \end{bmatrix}\begin{bmatrix} q_0 \\ q_1 \end{bmatrix} = \begin{bmatrix} q_0 + q_1/2 \\ q_0 - q_1/2 \end{bmatrix}$

$E\left\{[\mathbf{y}-\mathbf{x}]^{\mathrm{T}}[\mathbf{y}-\mathbf{x}]\right\} = E\left\{(q_0+q_1/2)^2\right\} + E\left\{(q_0-q_1/2)^2\right\} = 2\left[E\left\{q_0^{\,2}\right\} + E\left\{\left(\frac{q_1}{2}\right)^2\right\}\right]$

Consequently, the quantizer step size effecting the quantization error q_0 should be half of that relating to q_1.

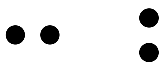
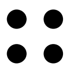
e)



$\mathbf{T}(4) = \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

Problem 5.1

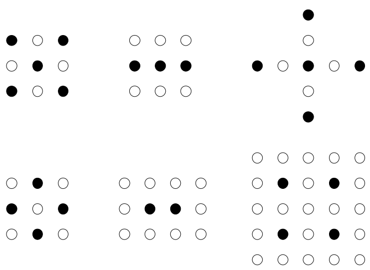
a) i) 20 ii) 15 iii) 25 iv) 20

b) iii) Two different root signals:  iv): 

Problem 5.2

Filter mask

Root signal



Problem 5.3

Assumption of constant boundary extension (4.9) for values outside of the marked bounding rectangles :

a) $\mathbf{Y} = \begin{bmatrix} 10 & 10 & 10 & 10 & 20 & 20 & 20 \\ 10 & 10 & 10 & 10 & 20 & 20 & 20 \\ 10 & 10 & 10 & 10 & 20 & 20 & 20 \\ 10 & 10 & 10 & 10 & 20 & 20 & 20 \\ 10 & 10 & 10 & 10 & 20 & 20 & 20 \end{bmatrix}$

b) $\mathbf{Y} = \begin{bmatrix} 10 & 10 & 20 & 20 & 20 & 20 & 20 \\ 10 & 10 & 20 & 20 & 20 & 20 & 20 \\ 10 & 10 & 20 & 20 & 20 & 20 & 20 \\ 10 & 10 & 10 & 20 & 20 & 20 & 20 \\ 10 & 10 & 10 & 20 & 20 & 20 & 20 \end{bmatrix}$

c) $\mathbf{Y} = \begin{bmatrix} 10 & 10 & 10 & 10 & 10 & 20 & 20 \\ 10 & 10 & 10 & 10 & 10 & 20 & 20 \\ 10 & 10 & 10 & 10 & 10 & 10 & 20 \\ 10 & 10 & 10 & 10 & 10 & 10 & 20 \\ 10 & 10 & 10 & 10 & 10 & 10 & 20 \end{bmatrix}$

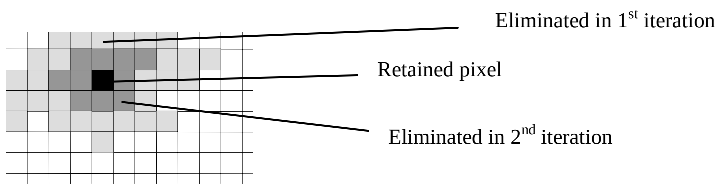
d) $\mathbf{Y} = \begin{bmatrix} 0 & 0 & 10 & 10 & 10 & 0 & 0 \\ 0 & 0 & 10 & 10 & 10 & 0 & 0 \\ 0 & 0 & 10 & 10 & 10 & 10 & 0 \\ 0 & 0 & 0 & 10 & 10 & 10 & 0 \\ 0 & 0 & 0 & 10 & 10 & 10 & 0 \end{bmatrix}$

e) $\mathbf{Y} = \begin{bmatrix} 10 & 10 & 10 & 10 & 20 & 20 & 20 \\ 10 & 10 & 10 & 10 & 20 & 20 & 20 \\ 10 & 10 & 10 & 10 & 20 & 20 & 20 \\ 10 & 10 & 10 & 10 & 10 & 20 & 20 \\ 10 & 10 & 10 & 10 & 10 & 20 & 20 \end{bmatrix}$

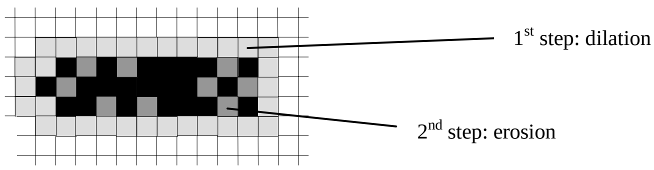
f) $\mathbf{Y} = \begin{bmatrix} 10 & 10 & 10 & 20 & 20 & 20 & 20 \\ 10 & 10 & 10 & 20 & 20 & 20 & 20 \\ 10 & 10 & 10 & 10 & 20 & 20 & 20 \\ 10 & 10 & 10 & 10 & 20 & 20 & 20 \\ 10 & 10 & 10 & 10 & 20 & 20 & 20 \end{bmatrix}$

Problem 5.4

a)

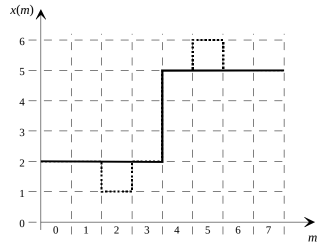


b)

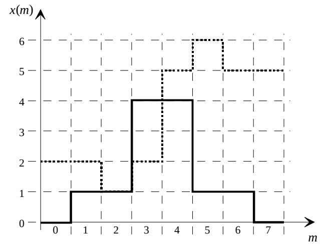


Effect : Filling of holes, straightening of edges

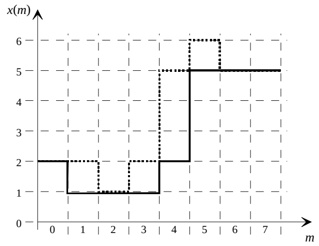
Problem 5.5



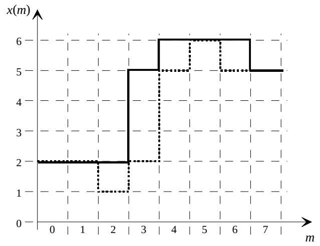
i) Median filter



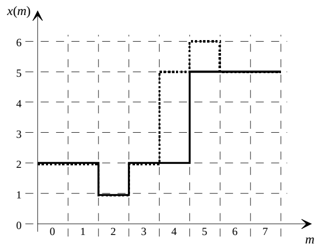
b) Maximum difference filter



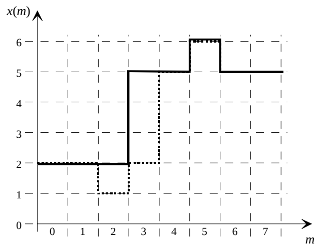
iii) Erosion filter



iv) Dilation filter



v) Opening filter



vi) Closing filter

Problem 5.6

$$x(\mathbf{n}) = \begin{cases} \alpha^{-1} \cdot y(\mathbf{n}) & \text{for} \quad y(\mathbf{n}) \le y_a \\ \beta^{-1} \cdot \left[y(\mathbf{n}) - y_a \right] + x_a & \text{for} \quad y_a \le y(\mathbf{n}) \le y_b \\ \gamma^{-1} \cdot \left[y(\mathbf{n}) - y_b \right] + x_b & \text{for} \quad y_b \le y(\mathbf{n}) \end{cases}$$

Problem 5.7

a) A=med[2,3,5]=3, B=med[3,4,5]=4, C=med[2,3,4]=3, D=med[2,4,5]=4

b) Parameters for bilinear interpolation: A – h=.25, v=.25 ; B – h=.25, v=.75.

$$\hat{A} = \frac{9}{16} \cdot 3 + \frac{3}{16} \cdot 2 + \frac{3}{16} \cdot 5 + \frac{1}{16} \cdot 4 = \frac{52}{16} = 3.25$$

$$\hat{B} = \frac{3}{16} \cdot 3 + \frac{1}{16} \cdot 2 + \frac{9}{16} \cdot 5 + \frac{3}{16} \cdot 4 = \frac{68}{16} = 4.25$$

Deviation between median and linear interpolation is .25 in both cases.

Problem 5.8

a) $$x(r') = c(m)\cdot\frac{r'^2}{2} + c(m-1)\cdot\left[-(r'+1)^2 + 3(r'+1) - 3/2\right] + c(m-2)\cdot\frac{(3-(r'+2))^2}{2}$$
$$= \frac{1}{2}\left[c(m)\cdot\frac{r'^2}{2} + c(m-1)\cdot\left(-2r'^2 + 2r' + 1\right) + c(m-2)\cdot\left(1 - 2r' + r'^2\right)\right]$$
$$= \frac{1}{2}\begin{bmatrix} r'^2 & r' & 1\end{bmatrix}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 1 & 0\end{bmatrix}\begin{bmatrix} c(m-2) \\ c(m-1) \\ c(m)\end{bmatrix}$$

b) with $r'=0$: $\hat{x}\left[r(m)\right] = \frac{1}{2}\left[c(m-1) + c(m-2)\right]$
$$\Rightarrow \begin{bmatrix} c(0) \\ c(1) \\ c(2) \\ \vdots \\ \\ c(M-1)\end{bmatrix} = 2\cdot\begin{bmatrix} 1 & 0 & 0 & & \cdots & 0 \\ 1 & 1 & 0 & 0 & & \vdots \\ 0 & 1 & 1 & 0 & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & \\ \vdots & & 0 & 1 & 1 & 0 \\ 0 & \cdots & & 0 & 1 & 1\end{bmatrix}^{-1}\cdot\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ \\ x(M-1)\end{bmatrix}$$

Problem 7.1

Angle between two K-dimensional vectors **x** and **y** :

$$\cos\varphi = \frac{x_1 y_1 + x_2 y_2 + \ldots + x_K y_K}{\sqrt{x_1{}^2 + x_2{}^2 + \ldots + x_K{}^2}\cdot\sqrt{y_1{}^2 + y_2{}^2 + \ldots + y_K{}^2}}$$

a) $$\varphi(Y, C_b) = \arccos\frac{-0.299\cdot 0.169 - 0.587\cdot 0.331 + 0.114\cdot 0,5}{\sqrt{0.299^2 + 0.587^2 + 0.114^2}\cdot\sqrt{0.169^2 + 0.331^2 + 0.5^2}} = 116.8^\circ$$
$$\varphi(Y, C_r) = \arccos\frac{0.299\cdot 0.5 - 0.587\cdot 0.419 - 0.114\cdot 0.081}{\sqrt{0.299^2 + 0.587^2 + 0.114^2}\cdot\sqrt{0.5^2 + 0.419^2 + 0.081^2}} = 103.9^\circ$$
$$\varphi(C_b, C_r) = \arccos\frac{-0.169\cdot 0.5 + 0.331\cdot 0.419 - 0.5\cdot 0.081}{\sqrt{0.169^2 + 0.331^2 + 0.5^2}\cdot\sqrt{0.5^2 + 0.419^2 + 0.081^2}} = 88.08^\circ$$

b) $$\varphi(X, Y) = \arccos\frac{0.607\cdot 0.299 + 0.174\cdot 0.587 + 0.200\cdot 0.114}{\sqrt{0.607^2 + 0.174^2 + 0.200^2}\cdot\sqrt{0.299^2 + 0.587^2 + 0.114^2}} = 46.2^\circ$$
$$\varphi(X, Z) = \arccos\frac{0.607\cdot 0.000 + 0.174\cdot 0.066 + 0.200\cdot 1.116}{\sqrt{0.607^2 + 0.174^2 + 0.200^2}\cdot\sqrt{0.000^2 + 0.066^2 + 1.116^2}} = 71.52^\circ$$
$$\varphi(Y, Z) = \arccos\frac{0.299\cdot 0.000 + 0.587\cdot 0.066 + 0.114\cdot 1.116}{\sqrt{0.299^2 + 0.587^2 + 0.114^2}\cdot\sqrt{0.000^2 + 0.066^2 + 1.116^2}} = 77.17^\circ$$

c) $$\varphi(I, K) = \arccos\frac{-\frac{1}{3}\cdot\frac{1}{\sqrt{6}} - \frac{1}{3}\cdot\frac{1}{\sqrt{6}} + \frac{1}{3}\cdot\frac{2}{\sqrt{6}}}{\sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2}\cdot\sqrt{\frac{1}{6} + \frac{1}{6} + \frac{4}{6}}} = 90^\circ$$

$$\varphi(I,L)=\arccos\frac{\frac{1}{3}\cdot\frac{1}{\sqrt{6}}-\frac{1}{3}\cdot\frac{1}{\sqrt{6}}+\frac{1}{3}\cdot 0}{\sqrt{\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^2}\cdot\sqrt{\frac{1}{6}+\frac{1}{6}+0}}=90^{\mathrm{o}}$$

$$\varphi(K,L)=\arccos\frac{-\frac{1}{\sqrt{6}}\cdot\frac{1}{\sqrt{6}}+\frac{1}{\sqrt{6}}\cdot\frac{1}{\sqrt{6}}+\frac{2}{\sqrt{6}}\cdot 0}{\sqrt{\frac{1}{6}+\frac{1}{6}+\frac{4}{6}}\cdot\sqrt{\frac{1}{6}+\frac{1}{6}+0}}=90^{\mathrm{o}}$$

Interpretation: Only for the I,K,L transform, basis vectors are perpendicular, however the transform is not orthogonal in strict sense, as the vectors have different Euclidean norms. For Y,C_r,C_b, the axes between chrominance components are approximately perpendicular. For X,Y,Z, the angles between the component axes are significantly lower than 90^0, which effects a limitation of the color variations that can be expressed.

Problem 7.2

a) If S=0 : $R=G=B=V\cdot A_{max}$

else :

If $0\leq H\leq\pi/3$: $R=MAX=V\cdot A_{max}$; $B=MIN=(1\text{-}S)\cdot R$; $G=H\cdot\frac{3}{\pi}\cdot S\cdot R+B$;

If $\pi/3\leq H\leq\pi$: $G=MAX=V\cdot A_{max}$; $B-R=\left(H\cdot\frac{3}{\pi}-2\right)\cdot(G-MIN)$;

If $\pi/3\leq H\leq 2\pi/3$: $B=MIN=(1\text{-}S)\cdot G$; $R=\left(2-H\cdot\frac{3}{\pi}\right)\cdot S\cdot G+B$;

If $2\pi/3\leq H\leq\pi$: $R=MIN=(1\text{-}S)\cdot G$; $B=\left(H\cdot\frac{3}{\pi}-2\right)\cdot S\cdot G+R$;

If $\pi\leq H\leq 5\pi/3$: $B=MAX=V\cdot A_{max}$; $R-G=\left(H\cdot\frac{3}{\pi}-4\right)\cdot(B-MIN)$;

If $\pi\leq H\leq 4\pi/3$: $R=MIN=(1\text{-}S)\cdot B$; $G=\left(4-H\cdot\frac{3}{\pi}\right)\cdot S\cdot B+R$;

If $4\pi/3\leq H\leq 5\pi/3$: $G=MIN=(1\text{-}S)\cdot B$; $R=\left(H\cdot\frac{3}{\pi}-4\right)\cdot S\cdot B+G$;

If $5\pi/3\leq H\leq 2\pi$: $R=MAX=V\cdot A_{max}$; $G=MIN=(1\text{-}S)\cdot R$; $B=\left(6-H\cdot\frac{3}{\pi}\right)\cdot S\cdot R+G$;

b) S=1, V=0.5 Yellow : $H=\pi/3 \Rightarrow R=0.5\cdot A_{max}$; $G=0.5\cdot A_{max}$; B=0

Cyan : $H=\pi\Rightarrow R=0$; $G=0.5\cdot A_{max}$; $B=0.5\cdot A_{max}$

Magenta : $H=5\pi/3 \Rightarrow R=0.5\cdot A_{max}$; G=0 ; $B=0.5\cdot A_{max}$

S=0.5, V=0.5 Yellow : $H=\pi/3 \Rightarrow R=0.5\cdot A_{max}$; $G=0.5\cdot A_{max}$; $B=0.25\cdot A_{max}$

Cyan : $H=\pi\Rightarrow R=0.25\cdot A_{max}$; $G=0.5\cdot A_{max}$; $B=0.5\cdot A_{max}$

Magenta : $H=5\pi/3 \Rightarrow R=0.5\cdot A_{max}$; $G=0.25\cdot A_{max}$; $B=0.5\cdot A_{max}$

c) According to results of b) or according to the cylinder-shaped color space (Fig. 7.2), the distance from the origin (black) is equal for any color tones of equal V and S values. For R,G,B according to (7.4), performing normalization by A_{max} :

S=1, V=0.5 : $d=\sqrt{0.5^2+0.5^2+0^2}=\sqrt{0.5}=\frac{\sqrt{2}}{2}\approx 0.7071$

S=0.5, *V*=0.5 : $d=\sqrt{0.5^2+0.5^2+0.25^2}=\sqrt{0.5625}=0.75$
For *H,S,V* according to (7.7)

S=1, *V*=0,5 : $d=\sqrt{\dfrac{0.5^2+1^2\cdot\left(\cos^2 H+\sin^2 H\right)}{5}}=\sqrt{\dfrac{1.25}{5}}=\sqrt{0.25}=0.5$

S=0,5, *V*=0,5 : $d=\sqrt{\dfrac{0{,}5^2+0{,}5^2\cdot\left(\cos^2 H+\sin^2 H\right)}{5}}=\sqrt{\dfrac{0{,}5}{5}}=\sqrt{0{,}10}\approx 0{,}316$

The color value of higher saturation has a larger distance from the origin (black) in the *H,S,V* color space. This is justified, as a "pure" color tone is perceived (even when having lower brightness) as more different from a black color.

Problem 7.3

a) $D_1=R-I \quad ; \quad D_2=B-I$

b) The basis vectors related to D_1 and D_2 are not orthogonal :
$\frac{2}{3}\cdot\left(-\frac{1}{3}\right)+\left(-\frac{1}{3}\right)\cdot\left(-\frac{1}{3}\right)+\left(-\frac{1}{3}\right)\cdot\frac{2}{3}=-\frac{1}{3}\neq 0$

c) From a) : $R=I+D_1 \quad ; \quad B=I+D_2 \quad ; \quad G=3I-R-B=I-D_1-D_2$

$$\begin{bmatrix}R\\G\\B\end{bmatrix}=\begin{bmatrix}1&1&0\\1&-1&-1\\1&0&1\end{bmatrix}\cdot\begin{bmatrix}I\\D_1\\D_2\end{bmatrix}$$

d) After transformation into ID_1D_2 color space :
$\mathbf{f}'_{A,1}=[0,0,0]^T$; $\mathbf{f}'_{B,1}=[1,0,0]^T$; $\mathbf{f}'_{A,2}=[1/3,2/3,-1/3]^T$; $\mathbf{f}'_{B,2}=[1/3,-1/3,2/3]^T$

RGB : $d_2(\mathbf{f}_{A,1},\mathbf{f}_{B,1})=\sqrt{1^2+1^2+1^2}=\sqrt{3} \quad ; \quad d_2(\mathbf{f}_{A,2},\mathbf{f}_{B,2})=\sqrt{1^2+0^2+(-1)^2}=\sqrt{2}$

ID_1D_2 : $d_2(\mathbf{f}_{A,1},\mathbf{f}_{B,1})=\sqrt{1^2+0^2+0^2}=1; \; d_2(\mathbf{f}_{A,2},\mathbf{f}_{B,2})=\sqrt{0^2+1^2+(-1)^2}=\sqrt{2}$

Pair 1 has pure intensity difference, pair 2 pure color difference; it can be concluded that the intensity difference is weighted stronger in case of *RGB*.

e) $\mu_I=\frac{1}{3}\cdot 5+\frac{1}{3}\cdot 8+\frac{1}{3}\cdot 2=5$; $\sigma_I{}^2=\left(\frac{1}{3}\right)^2\cdot 4+\left(\frac{1}{3}\right)^2\cdot 3+\left(\frac{1}{3}\right)^2\cdot 2=1$

Problem 7.4

a) It is only necessary to add the co-occurence values relating to the last column and row:

$$\mathbf{C}_\Delta{}^{(1,0)}=\begin{bmatrix}1&2&1\\3&1&0\\1&0&0\end{bmatrix};\;\mathbf{C}_\Delta{}^{(0,1)}=\begin{bmatrix}3&2&0\\2&0&1\\0&1&0\end{bmatrix};\;\mathbf{C}_\Delta{}^{(1,1)}=\begin{bmatrix}3&1&0\\1&1&1\\1&1&0\end{bmatrix}$$

This gives

$$\mathbf{C}^{(1,0)}=\begin{bmatrix}3&3&1\\2&3&1\\1&1&1\end{bmatrix}+\mathbf{C}_\Delta{}^{(1,0)}=\begin{bmatrix}4&5&2\\5&4&1\\2&1&1\end{bmatrix};\;\mathbf{C}^{(0,1)}=\begin{bmatrix}1&4&1\\4&1&2\\1&2&0\end{bmatrix}+\mathbf{C}_\Delta{}^{(0,1)}=\begin{bmatrix}4&6&1\\6&1&3\\1&3&0\end{bmatrix};$$

$$\mathbf{C}^{(1,1)}=\begin{bmatrix}4&2&1\\2&3&2\\0&2&0\end{bmatrix}+\mathbf{C}_\Delta{}^{(1,1)}=\begin{bmatrix}7&3&1\\3&4&3\\1&3&0\end{bmatrix}$$

b) The image matrix is a cyclic-shifted (shift by one pixel towards right and bottom) version of (7.18): Therefore, the co-occurrence matrices do not change.

$$\mathbf{X}_{cyc}=\begin{bmatrix}0&0&0&1&2\\1&1&0&1&1\\2&2&1&0&0\\1&1&0&2&0\\0&0&1&0&1\end{bmatrix}\begin{bmatrix}0\\1\\2\\1\\0\end{bmatrix} \qquad \mathbf{X}_{cyc,2}=\begin{bmatrix}2\\1\\0\\0\\1\end{bmatrix}\begin{bmatrix}0&0&0&1&2\\1&1&0&1&1\\2&2&1&0&0\\1&1&0&2&0\\0&0&1&0&1\end{bmatrix}$$

$$\begin{bmatrix}0&0&0&1&2\end{bmatrix}\begin{bmatrix}0\end{bmatrix} \qquad \begin{bmatrix}1\end{bmatrix}\begin{bmatrix}0&0&1&0&1\end{bmatrix}$$

c) $$\mathbf{P}^{(1,0)}=\frac{1}{25}\begin{bmatrix}4&5&2\\5&4&1\\2&1&1\end{bmatrix}\;;\;\mathbf{P}^{(0,1)}=\frac{1}{25}\begin{bmatrix}4&6&1\\6&1&3\\1&3&0\end{bmatrix}\;;\;\mathbf{P}^{(1,1)}=\frac{1}{25}\begin{bmatrix}7&3&1\\3&4&3\\1&3&0\end{bmatrix}$$

$\Rightarrow$ for $\mathbf{P}^{(1,1)}$ according to (7.22) with q=2:

$$\sum_i\sum_j (x_i-x_j)^2 P_{i,j}^{(1,1)}=\frac{1}{25}\left(7\cdot 0^2+3\cdot 1^2+1\cdot 2^2+3\cdot 1^2+4\cdot 0^2+3\cdot 1^2+1\cdot 2^2+3\cdot 1^2+0\cdot 0^2\right)=\frac{20}{25}=0.8$$

according to (7.24) with $\log_2$:

$$-\sum_i\sum_j P_{i,j}^{(k,l)}\cdot\log_2 P_{i,j}^{(1,1)}=0.514+0.367+0.186+0.367+0.423+0.367+0.186+0.367+0=2.777$$

according to (7.25):

$$\sum_i\sum_j\left(P_{i,j}^{(1,1)}\right)^2=\frac{1}{25^2}\left(7^2+3^2+1^2+3^2+4^2+3^2+1^2+3^2+0^2\right)=\frac{103}{625}=0.1648$$

Texture 1: Image with equality of entries: $\mathbf{P}^{(1,1)}=\frac{1}{9}\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$

$\Rightarrow$ criterion (7.22) =12/9; criterion (7.24) =$\log_2$ (1/9)=3.1699; criterion (7.25) =1/9

Texture 2: Image with constant gray level (e.g. level 1): $\mathbf{P}^{(1,1)}=\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}$

$\Rightarrow$ criterion (7.22) =0; criterion (7.24) =0; criterion (7.25) =1

Comparison texture 1: $d=|0.8-0|+|2.777-0|+|0.1648-1|=4.4122$

Comparison texture 2: $d=|0.8-1.333|+|2.777-3.1699|+|0.1648-0.1111|=.9792$

Problem 7.5

a) $P(0)=\dfrac{P^{(k,l)}(0,0)+P^{(k,l)}(0,1)}{P^{(k,l)}(0,0)+P^{(k,l)}(0,1)+P^{(k,l)}(1,0)+P^{(k,l)}(1,1)}=0.5 \quad \Rightarrow P(1)=1-P(0)=0.5$

b) $\mu_b=0.5\cdot 0+0.5\cdot 1=0.5 \quad ; \quad {\sigma_b}^2=0.5\cdot 0^2+0.5\cdot 1^2-{\mu_b}^2=0.25$

c) $r_{bb}(1,0)=0\cdot 0\cdot P^{(1,0)}(0,0)+0\cdot 1\cdot P^{(1,0)}(0,1)+1\cdot 0\cdot P^{(1,0)}(1,0)+1\cdot 1\cdot P^{(1,0)}(1,1)=0.45$

$r_{bb}(0,1)=0\cdot 0\cdot P^{(0,1)}(0,0)+0\cdot 1\cdot P^{(0,1)}(0,1)+1\cdot 0\cdot P^{(0,1)}(1,0)+1\cdot 1\cdot P^{(0,1)}(1,1)=0.4$

$r'_{bb}(1,0)=r_{bb}(1,0)-{\mu_b}^2=0.2 \quad ; \quad r'_{bb}(0,1)=r_{bb}(0,1)-{\mu_b}^2=0.15$

$\rho'_{bb}(1,0)=\dfrac{r'_{bb}(1,0)}{{\sigma_b}^2}=0.8 \quad ; \quad \rho'_{bb}(0,1)=\dfrac{r'_{bb}(0,1)}{{\sigma_b}^2}=0.6$

d) For AR(1) model: A=$\rho'_{bb}(1,0)$=0.8; B=$\rho'_{bb}(0,1)$=0.6; C=0, as AR(1) is zero mean.

Problem 7.6

a) According to (2.29), the frequency coordinate system rotates uniformly with the spatial coordinate system (for alias-free sampled images, this would strictly be true when sampling distances are equal horizontally and vertically):

$$\begin{bmatrix}\tilde{\Omega}_1\\ \tilde{\Omega}_2\end{bmatrix}=\begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix}\cdot\begin{bmatrix}\Omega_1\\ \Omega_2\end{bmatrix}$$

$$\begin{aligned}S_{xx}(\tilde{\Omega}_1,\tilde{\Omega}_2)&=\frac{\sigma_x^{\,2}(1-\rho^2)}{1-2\cdot\rho\cdot\cos\sqrt{(\Omega_1\cdot\cos\alpha+\Omega_2\cdot\sin\alpha)^2+(\Omega_2\cdot\cos\alpha-\Omega_1\cdot\sin\alpha)^2}+\rho^2}\\ &=\frac{\sigma_x^{\,2}(1-\rho^2)}{1-2\cdot\rho\cdot\cos\sqrt{\Omega_1^{\,2}\cdot\left(\cos^2\alpha+\sin^2\alpha\right)+\Omega_2^{\,2}\cdot\left(\cos^2\alpha+\sin^2\alpha\right)}+\rho^2}\\ &=S_{xx}(\Omega_1,\Omega_2)\end{aligned}$$

As the autocovariance statistics of the isotropic model are invariant against rotations, the power spectrum does not change.

b) The modification of frequency coordinates in case of pure scaling results analogously with (2.25):

$$\begin{bmatrix}\tilde{m}\\ \tilde{n}\end{bmatrix}=\begin{bmatrix}0.5 & 0\\ 0 & 0.5\end{bmatrix}\cdot\begin{bmatrix}m\\ n\end{bmatrix}\Rightarrow\begin{bmatrix}\tilde{\Omega}_1\\ \tilde{\Omega}_2\end{bmatrix}=\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}\cdot\begin{bmatrix}\Omega_1\\ \Omega_2\end{bmatrix}$$

$$\begin{bmatrix}\tilde{m}\\ \tilde{n}\end{bmatrix}=\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}\cdot\begin{bmatrix}m\\ n\end{bmatrix}\Rightarrow\begin{bmatrix}\tilde{\Omega}_1\\ \tilde{\Omega}_2\end{bmatrix}=\begin{bmatrix}0.5 & 0\\ 0 & 0.5\end{bmatrix}\cdot\begin{bmatrix}\Omega_1\\ \Omega_2\end{bmatrix}$$

$\Rightarrow$ for scaling 0.5:

$$\begin{aligned}S_{xx}(\tilde{\Omega}_1,\tilde{\Omega}_2)&=\frac{\sigma_x^{\,2}(1-\rho^2)}{1-2\cdot\rho\cdot\cos\sqrt{4\Omega_1^{\,2}+4\Omega_2^{\,2}}+\rho^2}\cdot\left(\det\begin{bmatrix}0.5 & 0\\ 0 & 0.5\end{bmatrix}\right)^2\\ &=\frac{\frac{1}{16}\sigma_x^{\,2}(1-\rho^2)}{1-2\cdot\rho\cdot\cos\left(2\cdot\sqrt{\Omega_1^{\,2}+\Omega_2^{\,2}}\right)+\rho^2}=\frac{1}{16}S_{xx}\left(2\Omega_1,2\Omega_2\right)\end{aligned}$$

$\Rightarrow$ for scaling 2:

$$\begin{aligned}S_{xx}(\tilde{\Omega}_1,\tilde{\Omega}_2)&=\frac{\sigma_x^{\,2}(1-\rho^2)}{1-2\cdot\rho\cdot\cos\sqrt{0.25\Omega_1^{\,2}+0.25\Omega_2^{\,2}}+\rho^2}\cdot\left(\det\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}\right)^2\\ &=\frac{16\cdot\sigma_x^{\,2}(1-\rho^2)}{1-2\cdot\rho\cdot\cos\left(0.5\cdot\sqrt{\Omega_1^{\,2}+\Omega_2^{\,2}}\right)+\rho^2}=16\cdot S_{xx}\left(\frac{\Omega_1}{2},\frac{\Omega_2}{2}\right)\end{aligned}$$

Problem 7.7

a)

$$\mathbf{Y}_h=\frac{1}{3}\begin{bmatrix}X & X & X & X & X & X\\ X & 0 & 5 & 5 & 5 & X\\ X & 5 & 10 & 10 & 5 & X\\ X & 5 & 5 & 5 & 0 & X\\ X & 10 & 5 & 0 & 0 & X\\ X & X & X & X & X & X\end{bmatrix}\qquad \mathbf{Y}_v=\frac{1}{3}\begin{bmatrix}X & X & X & X & X & X\\ X & 0 & 5 & 15 & 10 & X\\ X & 5 & 10 & 10 & 5 & X\\ X & 10 & 15 & 5 & 0 & X\\ X & 10 & 10 & 0 & 0 & X\\ X & X & X & X & X & X\end{bmatrix}$$

$$\mathbf{Y}_{d^+} = \frac{1}{3}\begin{bmatrix} X & X & X & X & X & X \\ X & 0 & 0 & 5 & 5 & X \\ X & 0 & 0 & 0 & 0 & X \\ X & 5 & 5 & 0 & 0 & X \\ X & 0 & 5 & 0 & 0 & X \\ X & X & X & X & X & X \end{bmatrix} \qquad \mathbf{Y}_{d^-} = \frac{1}{3}\begin{bmatrix} X & X & X & X & X & X \\ X & 0 & 5 & 15 & 10 & X \\ X & 5 & 15 & 15 & 5 & X \\ X & 10 & 15 & 5 & 0 & X \\ X & 15 & 10 & 0 & 0 & X \\ X & X & X & X & X & X \end{bmatrix}$$

b)

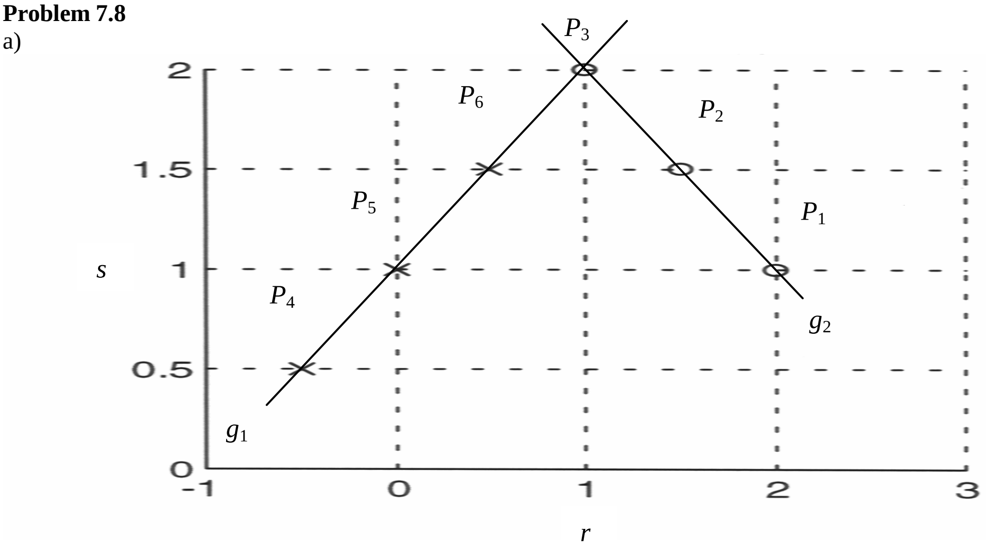
$$\mathbf{Y} = \frac{1}{3}\begin{bmatrix} X & X & X & X & X & X \\ X & 0 & 5 & 15 & 10 & X \\ X & 5 & 15 & 15 & 5 & X \\ X & 10 & 15 & 5 & 0 & X \\ X & 15 & 10 & 0 & 0 & X \\ X & X & X & X & X & X \end{bmatrix} \quad \text{with } 10/3<\Theta\leq 15/3: \quad \mathbf{B} = \begin{bmatrix} X & X & X & X & X & X \\ X & 0 & 0 & \mathbf{1} & 0 & X \\ X & 0 & \mathbf{1} & \mathbf{1} & 0 & X \\ X & 0 & \mathbf{1} & 0 & 0 & X \\ X & \mathbf{1} & 0 & 0 & 0 & X \\ X & X & X & X & X & X \end{bmatrix}$$

c)

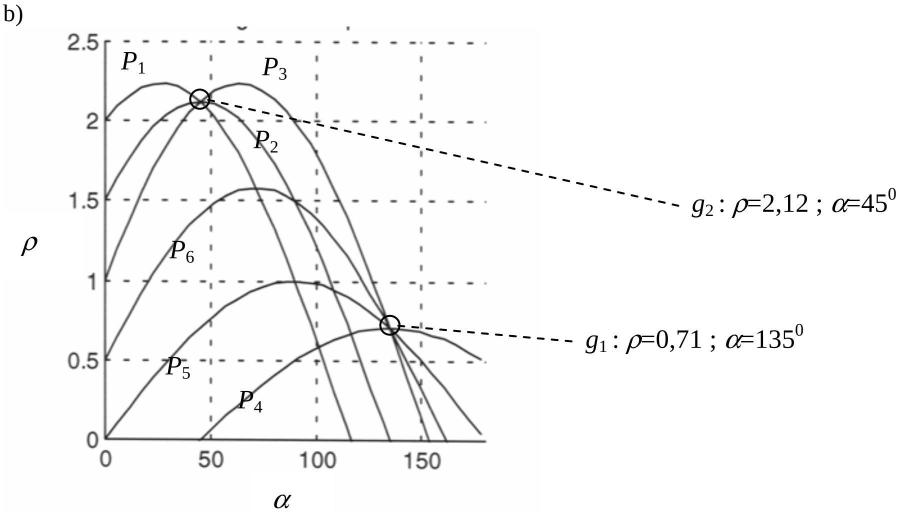
$$\mathbf{Y} = \frac{1}{4}\begin{bmatrix} X & X & X & X & X & X \\ X & 0 & 0 & -10 & 5 & X \\ X & 0 & -10 & 10 & 0 & X \\ X & -5 & 10 & 0 & 0 & X \\ X & -10 & 5 & 0 & 0 & X \\ X & X & X & X & X & X \end{bmatrix} \qquad \mathbf{Y}_{Abw} = \frac{1}{4}\begin{bmatrix} X & X & X & X & X & X \\ X & 0 & 10 & \mathbf{20} & 15 & X \\ X & 10 & \mathbf{20} & \mathbf{20} & 5 & X \\ X & \mathbf{20} & \mathbf{20} & 10 & 0 & X \\ X & 15 & \mathbf{20} & 5 & 0 & X \\ X & X & X & X & X & X \end{bmatrix}$$

Problem 7.8

a)



b)



Example for computation of the intersection P_1/P_2:

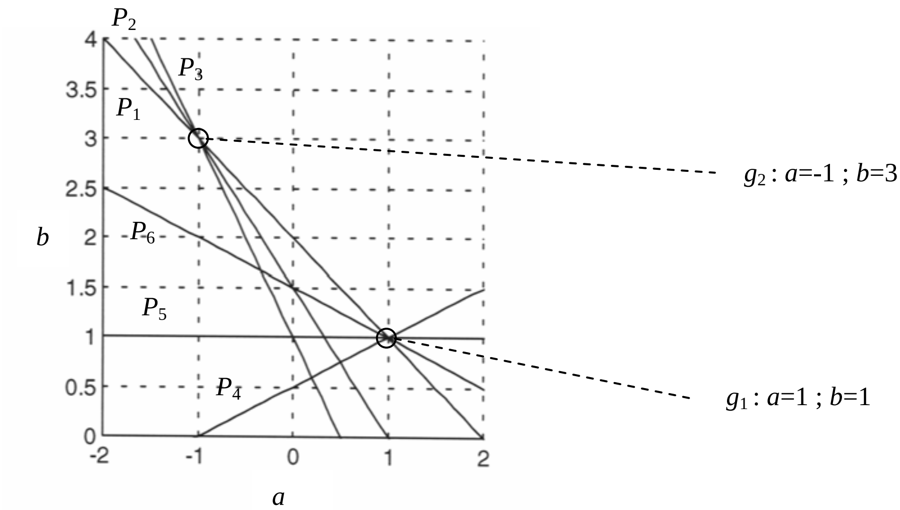
$P_1 : \rho = 2\cos\alpha + \sin\alpha \quad ; \quad P_2 : \rho = 1{,}5\cos\alpha + 1{,}5\sin\alpha$

$\Rightarrow 2\cos\alpha + \sin\alpha = 1{,}5\cos\alpha + 1{,}5\sin\alpha$

$\Rightarrow \cos\alpha = \sin\alpha \Rightarrow \alpha = \pm\frac{\pi}{4} \Rightarrow \rho = \pm 2\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \pm\frac{3}{2}\sqrt{2}$

The negative value of ρ cannot exist, such that a unique solution with positive α exists.

c) $s = a \cdot r + b \;\Rightarrow\; b = -r \cdot a + s$



d) The starting and ending points of lines are described by the respective intersecting graphs in the Hough space having highest and lowest slopes (for the case of Cartesian Hough space) or highest and lowest angles (for the case of polar Hough space).

Problem 7.9

a)

p=	0	1	2	3	4	5	6	7
$m(p)$	**1**	**2**	**3**	**4**	**3**	**2**	**1**	**1**
$n(p)$	**1**	**1**	**1**	**2**	**3**	**4**	**3**	**2**

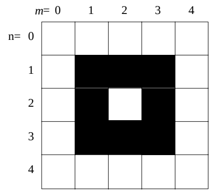
p=	0	1	2	3	4	5	6	7
$m(p)$	**1**	**2**	**3**	**3**	**3**	**2**	**1**	**1**
$n(p)$	**1**	**1**	**1**	**2**	**3**	**3**	**3**	**2**

b) Original contour:

$L = 4\left(\sqrt{2}+1\right) \approx 9{,}64$; a_1=a_2=4; A=11; ξ_K=11/16; ξ_F=16

Filtered contour (see Figure at right):

L=8; a_1=a_2=3; A=9; ξ_K=1; ξ_F=16



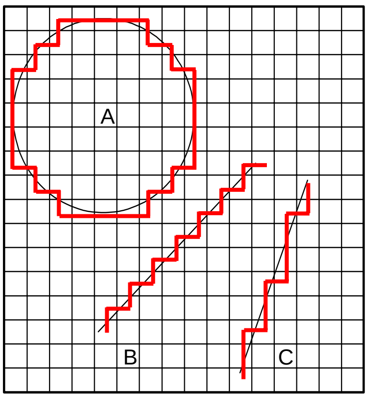
Problem 7.10

Lengths of continuous contours:

Shape A : Circle with radius =4, $\Rightarrow L_{orig}$=perimeter=$2\pi \cdot 4$=25.13 ;

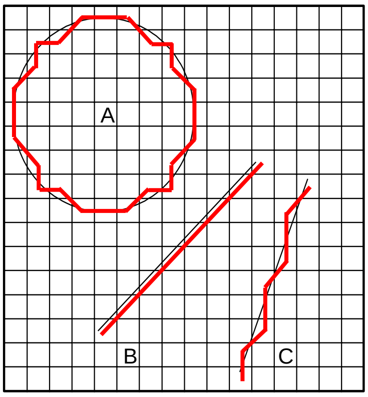
Shape B : $L_{orig} = \sqrt{7^2+7^2} \approx 9.9$; Shape C : $L_{orig} = \sqrt{3^2+8^2} \approx 8.54$

a)



The discrete contours are designed by connecting the centers of coordinate cells, such that the area between the continuous contour and the discrete approximation thereof becomes minimum. In the case of 4-neighbor ($\mathcal{N}_1^{(1)}$) connections, each connecting element has the length of 1. Shape A : L=32 ; Shape B : L=14 ; Shape C : L=12

b)



In the case of the 8-neighbor system ($\mathcal{N}_2^{(2)}$), horizontal and vertical connecting lines have a length of 1, while diagonal connecting lines have a length of $\sqrt{2}$.

Shape A: $L = 16 + 8 \cdot \sqrt{2} \approx 27.31$; B: $L = 7 \cdot \sqrt{2} \approx 9.9$; C: $L = 5 + 3 \cdot \sqrt{2} \approx 9.24$

In general, the approximation is better by using this system; the discrete contour is $\geq L_{orig}$ in any case, provided that the constraint stated above (minimization of area between contour and approximation) is observed.

Problem 7.11

$$\begin{aligned}
\rho^{(2,0)} &= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} (m - \overline{r})^2 \cdot x(m,n) \\
&= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} m^2 \cdot x(m,n) - 2 \cdot \overline{r} \cdot \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} m \cdot x(m,n) + \overline{r}^2 \cdot \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} x(m,n) \\
&= \mu^{(2,0)} - 2 \cdot \overline{r} \cdot \mu^{(1,0)} + \overline{r}^2 \cdot \mu^{(0,0)} = \mu^{(2,0)} - 2 \cdot \frac{\mu^{(1,0)}}{\mu^{(0,0)}} \cdot \mu^{(1,0)} + \left(\frac{\mu^{(1,0)}}{\mu^{(0,0)}} \right)^2 \cdot \mu^{(0,0)} \\
&= \mu^{(2,0)} - \frac{\left(\mu^{(1,0)}\right)^2}{\mu^{(0,0)}}
\end{aligned}$$

Similarly, $\rho^{(0,2)} = \mu^{(0,2)} - \dfrac{\left(\mu^{(0,1)}\right)^2}{\mu^{(0,0)}}$

Problem 7.12

a) The pixels of the skeleton are shown by gray shading.

	m= 0	1	2	3	4	5	6
n= 0							
1			1				
2			1	1		1	
3		1	2	2	1	1	
4			1	1		1	
5			1				
6							

b)

m / n =	0	1	2	3	4	5	6
$\Pi_v(m)$ =	0	1	5	3	1	3	0
$\Pi_h(n)$ =	0	1	3	5	3	1	0

c) The computation can be performed from the projection profiles, similar to the computation of statistical moments from histogram values (in principle, the projection profiles are histograms of coordinate distributions). Moments of orders 0 and 1, and subsequently the centroid coordinates of the shape are determined as follows:

$$\overline{r} = \frac{\mu^{(1,0)}}{\mu^{(0,0)}} = \frac{\sum_{m=0}^{M-1} m \cdot \Pi_v(m)}{\sum_{m=0}^{M-1} \Pi_v(m)} = \frac{0\cdot 0+1\cdot 1+2\cdot 5+3\cdot 3+4\cdot 1+5\cdot 3+6\cdot 0}{13} = \frac{39}{13} = 3$$

$$\overline{s} = \frac{\mu^{(0,1)}}{\mu^{(0,0)}} = \frac{\sum_{n=0}^{N-1} n \cdot \Pi_h(n)}{\sum_{n=0}^{N-1} \Pi_h(n)} = \frac{0\cdot 0+1\cdot 1+2\cdot 3+3\cdot 5+4\cdot 3+5\cdot 1+6\cdot 0}{13} = \frac{39}{13} = 3$$

d) The computation of moments $\mu^{(2,0)}$ and $\mu^{(0,2)}$ is performed similar to c):

$$\mu^{(2,0)} = \sum_{m=0}^{M-1} m^2 \cdot \Pi_v(m) = 0^2\cdot 0+1^2\cdot 1+2^2\cdot 5+3^2\cdot 3+4^2\cdot 1+5^2\cdot 3+6^2\cdot 0 = 139$$

$$\mu^{(0,2)} = \sum_{n=0}^{N-1} n^2 \cdot \Pi_h(n) = 0^2\cdot 0+1^2\cdot 1+2^2\cdot 3+3^2\cdot 5+4^2\cdot 3+5^2\cdot 1+6^2\cdot 0 = 131$$

Using the results from Problems 7.11 and 7.12c we get:

$$\rho^{(2,0)} = \mu^{(2,0)} - \frac{\left(\mu^{(1,0)}\right)^2}{\mu^{(0,0)}} = 139 - \frac{39^2}{13} = 22$$

$$\rho^{(0,2)} = \mu^{(0,2)} - \frac{\left(\mu^{(0,1)}\right)^2}{\mu^{(0,0)}} = 131 - \frac{39^2}{13} = 14$$

The moment $\mu^{(1,1)}$ can be computed only directly from the image (it plays a similar role as a covariance, which cannot be computed from the type of first-order distributions as the projection profiles are). The value computations for cases $x(m,n)\neq 0$ are performed in row-wise sequence in the subsequent formula:

$$\begin{aligned}\mu^{(1,1)} &= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} m \cdot n \cdot x(m,n) = \sum_{n=0}^{N-1} n \cdot \sum_{m=0}^{M-1} m \cdot x(m,n) \\ &= 1\cdot 2 + 2\cdot(2+3+5) + 3\cdot(1+2+3+4+5) + 4\cdot(2+3+5) + 5\cdot 2 = 117\end{aligned}$$

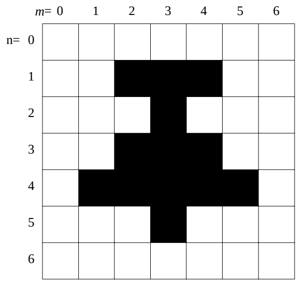
With (7.124) and the results of c) this gives

$$\rho^{(1,1)} = \mu^{(1,1)} - \frac{\mu^{(0,1)}\cdot\mu^{(1,0)}}{\mu^{(0,0)}} = 117 - \frac{39\cdot 39}{13} = 0 \Rightarrow \mathbf{\Gamma} = \begin{bmatrix} 22 & 0 \\ 0 & 14 \end{bmatrix}$$

$$m_1 = \rho^{(2,0)} + \rho^{(0,2)} = 22+14 = 36$$

$$m_2 = \sqrt{\left(\rho^{(2,0)} - \rho^{(0,2)}\right)^2 + 4\left(\rho^{(1,1)}\right)^2} = \sqrt{(22-14)^2} = 8$$

e) Rotated shape:



$$\tilde{\rho}^{(2,0)} = \rho^{(0,2)} \quad ; \quad \tilde{\rho}^{(0,2)} = \rho^{(2,0)}$$

$$\tilde{\mu}^{(1,1)} = 1\cdot(2+3+4)+2\cdot 3+3\cdot(2+3+4)+4\cdot(1+2+3+4+5)+5\cdot 3 = 117 \quad \Rightarrow \quad \tilde{\rho}^{(1,1)} = 0$$

$$\tilde{\mathbf{\Gamma}} = \begin{bmatrix} 14 & 0 \\ 0 & 22 \end{bmatrix}$$

$$\tilde{m}_1 = \tilde{\rho}^{(2,0)} + \tilde{\rho}^{(0,2)} = 14+22 = 36 = m_1 \qquad ;$$

$$\tilde{m}_2 = \sqrt{\left(\tilde{\rho}^{(2,0)} - \tilde{\rho}^{(0,2)}\right)^2 + 4\left(\tilde{\rho}^{(1,1)}\right)^2} = \sqrt{(14-22)^2} = 8 = m_2$$

Problem 7.13

a) Nine positions.

b) Shifted Versions $\mathbf{X}^{(k,l)}$ from $\mathbf{X}(o\text{-}1)$ are as follows; this shift is performed such that further processing can be made between frames $\mathbf{X}(o)$ and $\mathbf{X}(o\text{-}1)$ at same coordinate positions:

$$\mathbf{X}^{(1,1)} = \begin{bmatrix} 5&5&9&9&9&X\\ 5&5&9&9&9&X\\ 5&9&9&9&9&X\\ 9&9&9&9&9&X\\ 5&5&5&5&5&X\\ X&X&X&X&X&X \end{bmatrix} \quad \mathbf{X}^{(0,1)} = \begin{bmatrix} 5&5&5&9&9&9\\ 5&5&5&9&9&9\\ 5&5&9&9&9&9\\ 5&9&9&9&9&9\\ 5&5&5&5&5&5\\ X&X&X&X&X&X \end{bmatrix} \quad \mathbf{X}^{(-1,1)} = \begin{bmatrix} X&5&5&5&9&9\\ X&5&5&5&9&9\\ X&5&5&9&9&9\\ X&5&9&9&9&9\\ X&5&5&5&5&5\\ X&X&X&X&X&X \end{bmatrix}$$

$$\mathbf{X}^{(1,0)} = \begin{bmatrix} 5&5&9&9&9&X\\ 5&5&9&9&9&X\\ 5&5&9&9&9&X\\ 5&9&9&9&9&X\\ 9&9&9&9&9&X\\ 5&5&5&5&5&X \end{bmatrix} \quad \mathbf{X}^{(0,0)} = \begin{bmatrix} 5&5&5&9&9&9\\ 5&5&5&9&9&9\\ 5&5&5&9&9&9\\ 5&5&9&9&9&9\\ 5&9&9&9&9&9\\ 5&5&5&5&5&5 \end{bmatrix} \quad \mathbf{X}^{(-1,0)} = \begin{bmatrix} X&5&5&5&9&9\\ X&5&5&5&9&9\\ X&5&5&5&9&9\\ X&5&5&9&9&9\\ X&5&9&9&9&9\\ X&5&5&5&5&5 \end{bmatrix}$$

$$\mathbf{X}^{(1,-1)} = \begin{bmatrix} X&X&X&X&X&X\\ 5&5&9&9&9&X\\ 5&5&9&9&9&X\\ 5&5&9&9&9&X\\ 5&9&9&9&9&X\\ 9&9&9&9&9&X \end{bmatrix} \quad \mathbf{X}^{(0,-1)} = \begin{bmatrix} X&X&X&X&X&X\\ 5&5&5&9&9&9\\ 5&5&5&9&9&9\\ 5&5&5&9&9&9\\ 5&5&9&9&9&9\\ 5&9&9&9&9&9 \end{bmatrix} \quad \mathbf{X}^{(-1,-1)} = \begin{bmatrix} X&X&X&X&X&X\\ X&5&5&5&9&9\\ X&5&5&5&9&9\\ X&5&5&5&9&9\\ X&5&5&9&9&9\\ X&5&9&9&9&9 \end{bmatrix}$$

The related motion-compensated prediction error blocks $\mathbf{E}^{(k,l)}=\mathbf{X}(o)\text{-}\mathbf{X}^{(k,l)}$ are

$$\mathbf{E}^{(1,1)}=\begin{bmatrix} X & X & X & X & X & X \\ X & 0 & -4 & -2 & 0 & X \\ X & -4 & -2 & 0 & 0 & X \\ X & -2 & 0 & 0 & 0 & X \\ X & 0 & 0 & 0 & 0 & X \\ X & X & X & X & X & X \end{bmatrix}\quad \mathbf{E}^{(0,1)}=\begin{bmatrix} X & X & X & X & X & X \\ X & 0 & 0 & -2 & 0 & X \\ X & 0 & -2 & 0 & 0 & X \\ X & -2 & 0 & 0 & 0 & X \\ X & 0 & 0 & 0 & 0 & X \\ X & X & X & X & X & X \end{bmatrix}\quad \mathbf{E}^{(-1,1)}=\begin{bmatrix} X & X & X & X & X & X \\ X & 0 & 0 & -2 & 0 & X \\ X & 0 & -2 & 0 & 0 & X \\ X & -2 & 0 & 0 & 0 & X \\ X & 0 & 0 & 0 & 0 & X \\ X & X & X & X & X & X \end{bmatrix}$$

$$\mathbf{E}^{(1,0)}=\begin{bmatrix} X & X & X & X & X & X \\ X & 0 & -4 & -2 & 0 & X \\ X & 0 & -2 & 0 & 0 & X \\ X & -2 & 0 & 0 & 0 & X \\ X & -4 & -4 & -4 & -4 & X \\ X & X & X & X & X & X \end{bmatrix}\quad \mathbf{E}^{(0,0)}=\begin{bmatrix} X & X & X & X & X & X \\ X & 0 & 0 & -2 & 0 & X \\ X & 0 & 2 & 0 & 0 & X \\ X & 2 & 0 & 0 & 0 & X \\ X & -4 & -4 & -4 & -4 & X \\ X & X & X & X & X & X \end{bmatrix}\quad \mathbf{E}^{(-1,0)}=\begin{bmatrix} X & X & X & X & X & X \\ X & 0 & 0 & 2 & 0 & X \\ X & 0 & 2 & 4 & 0 & X \\ X & 2 & 4 & 0 & 0 & X \\ X & 0 & -4 & -4 & -4 & X \\ X & X & X & X & X & X \end{bmatrix}$$

$$\mathbf{E}^{(1,-1)}=\begin{bmatrix} X & X & X & X & X & X \\ X & 0 & -4 & -2 & 0 & X \\ X & 0 & -2 & 0 & 0 & X \\ X & 2 & 0 & 0 & 0 & X \\ X & -4 & -4 & -4 & -4 & X \\ X & X & X & X & X & X \end{bmatrix}\quad \mathbf{E}^{(0,-1)}=\begin{bmatrix} X & X & X & X & X & X \\ X & 0 & 0 & -2 & 0 & X \\ X & 0 & 2 & 0 & 0 & X \\ X & 2 & 4 & 0 & 0 & X \\ X & 0 & -4 & -4 & -4 & X \\ X & X & X & X & X & X \end{bmatrix}\quad \mathbf{E}^{(-1,-1)}=\begin{bmatrix} X & X & X & X & X & X \\ X & 0 & 0 & 2 & 0 & X \\ X & 0 & 2 & 4 & 0 & X \\ X & 2 & 4 & 4 & 0 & X \\ X & 0 & 0 & -4 & -4 & X \\ X & X & X & X & X & X \end{bmatrix}$$

Usage of the minimum squared error $\|\mathbf{E}\|^2$ as cost function indicates two equally optimum results (both with minimum cost =12) for (k,l)=(0,1) and (k,l)=(-1,1) :

$$\mathbf{K}^{(\sigma_e^2)}=\frac{1}{16}\begin{bmatrix} 44 & 12 & 12 \\ 92 & 76 & 92 \\ 92 & 76 & 92 \end{bmatrix}$$

Pixel-wise multiplication of $\mathbf{X}(o)$ and $\mathbf{X}^{(k,l)}$ is performed to compute the cross correlation:

$$\mathbf{R}^{(1,1)}=\begin{bmatrix} X & X & X & X & X & X \\ X & 25 & 45 & 63 & 81 & X \\ X & 45 & 63 & 81 & 81 & X \\ X & 63 & 81 & 81 & 81 & X \\ X & 25 & 25 & 25 & 25 & X \\ X & X & X & X & X & X \end{bmatrix}\quad \mathbf{R}^{(0,1)}=\begin{bmatrix} X & X & X & X & X & X \\ X & 25 & 25 & 63 & 81 & X \\ X & 25 & 63 & 81 & 81 & X \\ X & 63 & 81 & 81 & 81 & X \\ X & 25 & 25 & 25 & 25 & X \\ X & X & X & X & X & X \end{bmatrix}\quad \mathbf{R}^{(-1,1)}=\begin{bmatrix} X & X & X & X & X & X \\ X & 25 & 25 & 35 & 81 & X \\ X & 25 & 35 & 81 & 81 & X \\ X & 35 & 81 & 81 & 81 & X \\ X & 25 & 25 & 25 & 25 & X \\ X & X & X & X & X & X \end{bmatrix}$$

$$\mathbf{R}^{(1,0)}=\begin{bmatrix} X & X & X & X & X & X \\ X & 25 & 45 & 63 & 81 & X \\ X & 25 & 63 & 81 & 81 & X \\ X & 63 & 81 & 81 & 81 & X \\ X & 45 & 45 & 45 & 45 & X \\ X & X & X & X & X & X \end{bmatrix}\quad \mathbf{R}^{(0,0)}=\begin{bmatrix} X & X & X & X & X & X \\ X & 25 & 25 & 63 & 81 & X \\ X & 25 & 35 & 81 & 81 & X \\ X & 35 & 81 & 81 & 81 & X \\ X & 45 & 45 & 45 & 45 & X \\ X & X & X & X & X & X \end{bmatrix}\quad \mathbf{R}^{(-1,0)}=\begin{bmatrix} X & X & X & X & X & X \\ X & 25 & 25 & 35 & 81 & X \\ X & 25 & 35 & 45 & 81 & X \\ X & 35 & 45 & 81 & 81 & X \\ X & 25 & 45 & 45 & 45 & X \\ X & X & X & X & X & X \end{bmatrix}$$

$$\mathbf{R}^{(1,-1)}=\begin{bmatrix} X & X & X & X & X & X \\ X & 25 & 45 & 63 & 81 & X \\ X & 25 & 63 & 81 & 81 & X \\ X & 35 & 81 & 81 & 81 & X \\ X & 45 & 45 & 45 & 45 & X \\ X & X & X & X & X & X \end{bmatrix}\quad \mathbf{R}^{(0,-1)}=\begin{bmatrix} X & X & X & X & X & X \\ X & 25 & 25 & 63 & 81 & X \\ X & 25 & 35 & 81 & 81 & X \\ X & 35 & 45 & 81 & 81 & X \\ X & 25 & 45 & 45 & 45 & X \\ X & X & X & X & X & X \end{bmatrix}\quad \mathbf{R}^{(-1,-1)}=\begin{bmatrix} X & X & X & X & X & X \\ X & 25 & 25 & 35 & 81 & X \\ X & 25 & 35 & 45 & 81 & X \\ X & 35 & 45 & 45 & 81 & X \\ X & 25 & 25 & 81 & 81 & X \\ X & X & X & X & X & X \end{bmatrix}$$

The cost function indicates a less clear optimum for (k,l)=(1,0) (usage of cross covariance and normalization would give a more evident result):

$$\mathbf{K}^{(r_{xy})} = \frac{1}{16}\begin{bmatrix} 890 & 769 & 685 \\ 950 & 874 & 754 \\ 922 & 818 & 770 \end{bmatrix}$$

2. The shift by half a pixel using linear interpolation is equivalent with the following averaging operation:

$$\mathbf{X}^{(-\frac{1}{2},1)} = \mathbf{X}^{(0,1)} + \mathbf{X}^{(-1,1)} = \frac{1}{2}\left[\begin{bmatrix} 5 & 5 & 5 & 9 & 9 & 9 \\ 5 & 5 & 5 & 9 & 9 & 9 \\ 5 & 5 & 9 & 9 & 9 & 9 \\ 5 & 9 & 9 & 9 & 9 & 9 \\ 5 & 5 & 5 & 5 & 5 & 5 \\ X & X & X & X & X & X \end{bmatrix} + \begin{bmatrix} X & 5 & 5 & 5 & 9 & 9 \\ X & 5 & 5 & 5 & 9 & 9 \\ X & 5 & 5 & 9 & 9 & 9 \\ X & 5 & 9 & 9 & 9 & 9 \\ X & 5 & 5 & 5 & 5 & 5 \\ X & X & X & X & X & X \end{bmatrix}\right] = \begin{bmatrix} X & X & X & X & X & X \\ X & 5 & 5 & 7 & 9 & X \\ X & 5 & 7 & 9 & 9 & X \\ X & 7 & 9 & 9 & 9 & X \\ X & 5 & 5 & 5 & 5 & X \\ X & X & X & X & X & X \end{bmatrix}$$

Within the current block, the resultant matrix is exactly equal to the matrix **X**(o). The cost function using minimum squared difference will give the value of zero.

Problem 7.14

$$r' = a_1 \cdot r - a_2 \cdot s + t_1 \Rightarrow r' - r = (a_1 - 1)\cdot r - a_2 \cdot s + t_1$$
$$\Rightarrow k(m,n)\cdot R = (a_1 - 1)\cdot m \cdot R - a_2 \cdot n \cdot S + t_1$$
$$s' = a_2 \cdot r + a_1 \cdot s + t_2 \Rightarrow s' - s = a_2 \cdot r + (a_1 - 1)\cdot s + t_2$$
$$\Rightarrow l(m,n)\cdot S = a_2 \cdot m \cdot R + (a_1 - 1)\cdot n \cdot S + t_2$$

Normalization by sampling intervals:

$$k(m,n) = (a_1 - 1)\cdot m - a_2 \cdot \frac{S}{R}\cdot n + \frac{t_1}{R}$$
$$l(m,n) = a_2 \cdot \frac{R}{S}\cdot m + (a_1 - 1)\cdot n + \frac{t_2}{S}$$

Substitution into optical flow equation:

$$\left[(a_1 - 1)\cdot m - a_2 \cdot \frac{S}{R}\cdot n + \frac{t_1}{R}\right]\cdot x_r(m,n) + \left[a_2 \cdot \frac{R}{S}\cdot m + (a_1 - 1)\cdot n + \frac{t_2}{S}\right]\cdot x_s(m,n) = -x_t(m,n)$$

The resulting over-determined equation system is

$$\begin{bmatrix} m\cdot x_r(1) + n\cdot x_s(1) & m\cdot\frac{R}{S}\cdot x_s(1) - n\cdot\frac{S}{R}\cdot x_r(1) & \frac{1}{R}\cdot x_r(1) & \frac{1}{S}\cdot x_s(1) \\ m\cdot x_r(2) + n\cdot x_s(2) & m\cdot\frac{R}{S}\cdot x_s(2) - n\cdot\frac{S}{R}\cdot x_r(2) & \frac{1}{R}\cdot x_r(2) & \frac{1}{S}\cdot x_s(2) \\ \vdots & \vdots & \vdots & \vdots \\ m\cdot x_r(P) + n\cdot x_s(P) & m\cdot\frac{R}{S}\cdot x_s(P) - n\cdot\frac{S}{R}\cdot x_r(P) & \frac{1}{R}\cdot x_r(P) & \frac{1}{S}\cdot x_s(P) \end{bmatrix}\cdot\begin{bmatrix} a_1 - 1 \\ a_2 \\ t_1 \\ t_2 \end{bmatrix} = -\begin{bmatrix} x_t(1) \\ x_t(2) \\ \vdots \\ x_t(P) \end{bmatrix}$$

Problem 8.1

a) $$H(z_1) = \frac{1}{3}\left[z^1 + 1 + z^{-1}\right]$$

$$H(j\Omega_1) = \frac{1}{3}\left[e^{j\Omega_1} + 1 + e^{-j\Omega_1}\right] = \frac{1}{3}\left[1 + 2\cos\Omega_1\right]$$

b) $1+2\cos\Omega_1=0$ for $\cos\Omega_1=-1/2 \Rightarrow$ Zeros of transfer function at $\Omega_1=\pi\pm\pi/3$

Inverse filter : $H^I(j\Omega_1)=\dfrac{3}{1+2\cos\Omega_1}$

Pseudo inverse filter : $H^{PI}(j\Omega_1)=\begin{cases}\dfrac{3}{1+2\cos\Omega_1} & \text{for } \Omega_1\neq\pi\pm\dfrac{\pi}{3}\\ 0 \text{ for } \Omega_1=\pi\pm\dfrac{\pi}{3}\end{cases}$

Discrete frequency lines of DFT at $\Omega_1(u)=u\cdot\dfrac{2\pi}{M}$

$H^I(u)=\dfrac{3}{1+\cos\dfrac{2\pi u}{M}}$; critical cases are $\dfrac{u}{M}=\dfrac{1}{2}\pm\dfrac{1}{6}$ for integer u.

Inverse filtering unstable for u=10, u=20 with M=30; no instability for M=32.

c) Power spectrum of the attenuated signal $S_{yy}(\Omega_1)=A^2\cdot\dfrac{1}{9}\cdot\left[1+2\cos\Omega_1\right]^2$

Power spectrum of disturbing noise $S_{zz}(\Omega_1)=\dfrac{A^2}{4}$

Condition $A^2\cdot\dfrac{1}{9}\cdot\left[1+2\cos\Omega_1\right]^2\geq\dfrac{A^2}{4}$

$1+4\cos\Omega_1+4\cos^2\Omega_1\geq\dfrac{9}{4}\Rightarrow\cos^2\Omega_1+\cos\Omega_1-\dfrac{5}{16}\geq 0$

Limit case for $\cos\Omega_1=-\dfrac{1}{2}\pm\sqrt{\dfrac{1}{4}+\dfrac{5}{16}}=-\dfrac{1}{2}\pm\dfrac{3}{4}$, only the non-transcendent case with |cos|<1 is regarded here $\Rightarrow$ arccos(1/4)≈0.4196π:

$\Rightarrow H^{PI}(j\Omega_1)=\begin{cases}\dfrac{3}{1+2\cos\Omega_1} & \text{for } \Omega_1\leq 0.4196\pi\\ 0 \text{ for } \Omega_1>0.4196\pi\end{cases}$

d) Wiener filter:

$$H^I(j\Omega_1)=\frac{H^*(j\Omega_1)}{\left|H(j\Omega_1)\right|^2+\dfrac{S_{zz}(\Omega_1)}{S_{xx}(\Omega_1)}}=\frac{\dfrac{1}{3}\left[1+2\cos\Omega_1\right]}{\dfrac{1}{9}\left[1+2\cos\Omega_1\right]^2+\dfrac{A^2/4}{A^2}}=\frac{1+2\cos\Omega_1}{\dfrac{1}{3}\left[1+2\cos\Omega_1\right]^2+\dfrac{3}{4}}$$

Attenuation for Ω_1=0.4196π: $\dfrac{1+2\cdot\dfrac{1}{4}}{\dfrac{1}{3}\left[1+2\cdot\dfrac{1}{4}\right]^2+\dfrac{3}{4}}=1$

Problem 8.2

a) $H(j\Omega)=\frac{1}{4}\cdot\left(e^{j\Omega}+2+e^{-j\Omega}\right)=\frac{1}{2}\cdot\left(1+\cos\Omega\right)$

b) $H^{\mathrm{I}}(j\Omega)=\dfrac{1}{H(j\Omega)}=\dfrac{2}{1+\cos\Omega}$. Zero for $\cos\Omega=-1$, i.e. $\Omega=\pi$ (half of sampling frequency). No pseudo inverse filter necessary, if sampling conditions are observed.

c) $$H^{IW}(j\Omega)=\frac{\frac{1}{2}\left(1+\cos\Omega\right)}{\frac{1}{4}\left(1+\cos\Omega\right)^2+\frac{1}{4}\cdot\sin^2\Omega}=\frac{\frac{1}{2}\left(1+\cos\Omega\right)}{\frac{1}{4}+\frac{1}{2}\cos\Omega+\frac{1}{4}\cos^2\Omega+\frac{1}{4}\cdot\sin^2\Omega}=1$$

d) Substitution $|\Omega|=\sqrt{\Omega_1{}^2+\Omega_2{}^2}$: $H_{2D}(j\Omega_1,j\Omega_2)=\frac{1}{2}\cdot\left(1+\cos\sqrt{\Omega_1{}^2+\Omega_2{}^2}\right)$

e) Transfer function =0 for $\sqrt{\Omega_1{}^2+\Omega_2{}^2}=\pi$, pseudo inverse filter necessary:

$$H_{2D}{}^{IP}(j\Omega_1,j\Omega_2)=\begin{cases}\dfrac{2}{1+\cos\sqrt{\Omega_1{}^2+\Omega_2{}^2}}, & \text{if } \sqrt{\Omega_1{}^2+\Omega_2{}^2}\neq\pi\\ 0, \ \text{if } \sqrt{\Omega_1{}^2+\Omega_2{}^2}=\pi\end{cases}$$

Problem 8.3

a) Case i) $\mathbf{y}=\begin{bmatrix}\frac{1}{2}&\frac{1}{2}&0\\0&\frac{1}{2}&\frac{1}{2}\end{bmatrix}\cdot\begin{bmatrix}3\\9\\15\end{bmatrix}=\begin{bmatrix}6\\12\end{bmatrix}$ Case ii) $\mathbf{y}=\begin{bmatrix}1&0&0\\\frac{1}{3}&\frac{2}{3}&0\\0&\frac{2}{3}&\frac{1}{3}\\0&0&1\end{bmatrix}\cdot\begin{bmatrix}3\\9\\15\end{bmatrix}=\begin{bmatrix}3\\7\\11\\15\end{bmatrix}$

b) Case i) according to (8.20), $L>K$: $\mathbf{H}^{\mathrm{p}}=\mathbf{H}^{\mathrm{T}}\cdot\left(\mathbf{HH}^{\mathrm{T}}\right)^{-1}$

$$\left(\mathbf{HH}^{\mathrm{T}}\right)^{-1}=\left(\begin{bmatrix}\frac{1}{2}&\frac{1}{2}&0\\0&\frac{1}{2}&\frac{1}{2}\end{bmatrix}\begin{bmatrix}\frac{1}{2}&0\\\frac{1}{2}&\frac{1}{2}\\0&\frac{1}{2}\end{bmatrix}\right)^{-1}=\begin{bmatrix}\frac{1}{2}&\frac{1}{4}\\\frac{1}{4}&\frac{1}{2}\end{bmatrix}^{-1}=\begin{bmatrix}\frac{8}{3}&-\frac{4}{3}\\-\frac{4}{3}&\frac{8}{3}\end{bmatrix}$$

$$\Rightarrow\mathbf{H}^{\mathrm{p}}=\begin{bmatrix}\frac{1}{2}&0\\\frac{1}{2}&\frac{1}{2}\\0&\frac{1}{2}\end{bmatrix}\cdot\begin{bmatrix}\frac{8}{3}&-\frac{4}{3}\\-\frac{4}{3}&\frac{8}{3}\end{bmatrix}=\begin{bmatrix}\frac{4}{3}&-\frac{2}{3}\\\frac{2}{3}&\frac{2}{3}\\-\frac{2}{3}&\frac{4}{3}\end{bmatrix}$$

Case ii) according to (8.20), $L<K$: $\mathbf{H}^{\mathrm{p}}=\left(\mathbf{H}^{\mathrm{T}}\mathbf{H}\right)^{-1}\cdot\mathbf{H}^{\mathrm{T}}$

$$\left(\mathbf{H}^{\mathrm{T}}\mathbf{H}\right)^{-1}=\left(\begin{bmatrix}1&\frac{1}{3}&0&0\\0&\frac{2}{3}&\frac{2}{3}&0\\0&0&\frac{1}{3}&1\end{bmatrix}\begin{bmatrix}1&0&0\\\frac{1}{3}&\frac{2}{3}&0\\0&\frac{2}{3}&\frac{1}{3}\\0&0&1\end{bmatrix}\right)^{-1}=\begin{bmatrix}\frac{10}{9}&\frac{2}{9}&0\\\frac{2}{9}&\frac{8}{9}&\frac{2}{9}\\0&\frac{2}{9}&\frac{10}{9}\end{bmatrix}^{-1}=\begin{bmatrix}0.95&-0.25&0.05\\-0.25&1.25&-0.25\\0.05&-0.25&0.95\end{bmatrix}$$

$\Rightarrow$

$$\mathbf{H}^{\mathrm{p}}=\begin{bmatrix}0.95&-0.25&0.05\\-0.25&1.25&-0.25\\0.05&-0.25&0.95\end{bmatrix}\cdot\begin{bmatrix}1&\frac{1}{3}&0&0\\0&\frac{2}{3}&\frac{2}{3}&0\\0&0&\frac{1}{3}&1\end{bmatrix}=\begin{bmatrix}0.95&0.15&-0.15&0.05\\-0.25&0.75&0.75&-0.25\\0.05&-0.15&0.15&0.95\end{bmatrix}$$

c) Case i) $\hat{\mathbf{x}} = \begin{bmatrix} \frac{4}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{4}{3} \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 12 \end{bmatrix} = \begin{bmatrix} 0 \\ 12 \\ 12 \end{bmatrix}$

Case ii) $\hat{\mathbf{x}} = \begin{bmatrix} 0.95 & 0.15 & -0.15 & 0.05 \\ -0.25 & 0.75 & 0.75 & -0.25 \\ 0.05 & -0.15 & 0.15 & 0.95 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 7 \\ 11 \\ 15 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \\ 15 \end{bmatrix}$

d) Eigenvalues of $\mathbf{HH}^{\mathrm{T}}$ (cf. Problem 4.1) : λ_0=1/2+1/4=3/4 ; λ_1=1/2-1/4=1/4

$\Rightarrow$ Matrix of singular values (rank of $\mathbf{H}$ is R=2)

$$\mathbf{U}^{\mathrm{T}}\mathbf{HV} = \mathbf{\Lambda}^{(1/2)} = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$$

$$\mathbf{U}^{\mathrm{T}}\left[\mathbf{HH}^{\mathrm{T}}\right]\mathbf{U} = \mathbf{U}^{\mathrm{T}} \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix} \mathbf{U} = \mathbf{\Lambda}^{(K)} = \begin{bmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix} ;\ \mathbf{U} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

$$\mathbf{V}^{\mathrm{T}}\left[\mathbf{H}^{\mathrm{T}}\mathbf{H}\right]\mathbf{V} = \mathbf{V}^{\mathrm{T}} \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \mathbf{V} = \mathbf{\Lambda}^{(L)} = \begin{bmatrix} \frac{3}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 \end{bmatrix} ;\ \mathbf{V} = \begin{bmatrix} \frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{2} & 0 & -\frac{\sqrt{3}}{3} \\ \frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} \end{bmatrix}$$

Computation of estimate by generalized inverse from the first singular value:

$$\mathbf{H}_0{}^{\mathrm{g}} = (\lambda_0)^{-1/2}\,\mathbf{v}_0\mathbf{u}_0{}^{\mathrm{T}} = \frac{2}{\sqrt{3}} \cdot \begin{bmatrix} \frac{\sqrt{6}}{6} \\ \frac{\sqrt{3}}{2} \\ \frac{\sqrt{6}}{6} \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$\Rightarrow \hat{\mathbf{x}} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 12 \end{bmatrix} = \begin{bmatrix} 6 \\ 12.7 \\ 6 \end{bmatrix}$$

Problem 8.4

Mean and variance of the signal:

$$\mu_x = 0.2\cdot 0 + 0.8\cdot 1 = 0.8$$

$$\sigma_x{}^2 = \left[0.2\cdot 0^2 + 0.8\cdot 1^2\right] - \mu_x{}^2 = 0.8 - 0.64 = 0.16$$

The inverse "autocovariance matrices" of size 1x1 relate to variances of the signal and the noise,

$$\mathbf{R}_{xx}{}^{-1} = \frac{1}{\sigma_x{}^2} ;\quad \mathbf{R}_{zz}{}^{-1} = \frac{1}{\sigma_z{}^2}$$

a) Maximum likelihood criterion (8.38) : Minimization of $\Delta_{\mathrm{ML}} = \frac{[y-\hat{x}]^2}{2\sigma_z^{\ 2}}$ gives

	y=0.3	y=0.5	y=0.7
$\hat{x}=0$	0.45	1.25	2.45
$\hat{x}=1$	2.45	1.25	0.45

The decision for the value y=0.5 is not unique; the behavior is equivalent to a threshold decision with Θ=0.5

b) Maximum-a-posteriori criterion according to (8.45) : Minimization of

$$\Delta_{\mathrm{MAP}} = \frac{[y-\hat{x}]^2}{2\sigma_z^{\ 2}} + \frac{[\hat{x}-\mu_x]^2}{2\sigma_x^{\ 2}}$$

	y=0.3	y=0.5	y=0.7
$\hat{x}=0$	0.45+2=2.45	1.25+2=3.25	2.45+2
$\hat{x}=1$	2.45+0.125=2.575	1.25+0.125=1.375	0.45+0.125=0.575

The value y=0.5 is now clearly decided for $\hat{x}=1$; the behavior is equivalent to a threshold decision with

$$\frac{\Theta^2}{2\sigma_z^{\ 2}} + 2 = \frac{(\Theta-1)^2}{2\sigma_z^{\ 2}} + 0.125 \Rightarrow \Theta = 0.3125$$

Problem 8.5

a) $\mathbf{R}_{zz} = \mathbf{R}_{zz}^{\ -1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; $\mathbf{R}_{xx} = \frac{16}{7}\begin{bmatrix} 1 & 3/4 \\ 3/4 & 1 \end{bmatrix}$

$$\Rightarrow \mathbf{R}_{xx}^{\ -1} = \frac{7}{16} \cdot \frac{1}{(1-9/16)} \begin{bmatrix} 1 & -3/4 \\ -3/4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3/4 \\ -3/4 & 1 \end{bmatrix}$$

b) $\mathbf{y}-\hat{\mathbf{x}}_1 = [1 \quad -1]^{\mathrm{T}}$; $\mathbf{y}-\hat{\mathbf{x}}_2 = [-2 \quad 0]^{\mathrm{T}}$

$$\Delta_{\mathrm{ML}}(\hat{\mathbf{x}}_1) = \frac{1}{2}\cdot[1 \quad -1]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}[1 \quad -1]^{\mathrm{T}} = 1 \quad ; \quad \Delta_{\mathrm{ML}}(\hat{\mathbf{x}}_2) = \frac{1}{2}\cdot[-2 \quad 0]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}[-2 \quad 0]^{\mathrm{T}} = 2$$

This leads to the conclusion that the first hypothesis is better.

c)

$$\Delta_{\mathrm{MAP}}(\hat{\mathbf{x}}_1) = \frac{1}{2}\cdot[0 \quad 4]\begin{bmatrix} 1 & -3/4 \\ -3/4 & 1 \end{bmatrix}[0 \quad 4]^{\mathrm{T}} + \frac{1}{2}\cdot[1 \quad -1]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}[1 \quad -1]^{\mathrm{T}} = 8+1 = 9$$

$$\Delta_{\mathrm{MAP}}(\hat{\mathbf{x}}_2) = \frac{1}{2}\cdot[3 \quad 3]\begin{bmatrix} 1 & -3/4 \\ -3/4 & 1 \end{bmatrix}[3 \quad 3]^{\mathrm{T}} + \frac{1}{2}\cdot[-2 \quad 0]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}[-2 \quad 0]^{\mathrm{T}} = 2.25+2 = 4.25$$

Now, the second hypothesis is better, which can be explained by the correlated signal model.

d) The signal variance reciprocally influences the first term of the MAP estimate. Both hypotheses would be equivalent, if

$$\frac{8}{c}+1=\frac{2.25}{c}+2 \quad \Rightarrow c=5.75$$

For $\sigma_x^2 > c \cdot 16/7$, the first hypothesis is preferred by the MAP estimation: If higher variance of the signal is expected, the high deviation between both values in the vector appears more reasonable.

Problem 9.1

a) Class centroid vectors: $\mathbf{z}^{(1)}=c\cdot\begin{bmatrix}1.5\\1.5\end{bmatrix};\ \mathbf{z}^{(2)}=\begin{bmatrix}4.5\\3\end{bmatrix};\ i)\ \mathbf{z}=\begin{bmatrix}2.625\\1.875\end{bmatrix};\ ii)\ \mathbf{z}=\begin{bmatrix}3.75\\3\end{bmatrix}$

Covariance matrices:

$$\mathbf{C}_{mm}{}^{(1)}=\frac{1}{4}\cdot c^2\cdot\left(\begin{bmatrix}0.25 & 0.25\\0.25 & 0.25\end{bmatrix}+\begin{bmatrix}0.25 & -0.25\\-0.25 & 0.25\end{bmatrix}+\begin{bmatrix}0.25 & -0.25\\-0.25 & 0.25\end{bmatrix}+\begin{bmatrix}0.25 & 0.25\\0.25 & 0.25\end{bmatrix}\right)$$
$$=c^2\cdot\begin{bmatrix}0.25 & 0\\0 & 0.25\end{bmatrix}$$

$$\mathbf{C}_{mm}{}^{(2)}=\frac{1}{4}\cdot\left(\begin{bmatrix}0.25 & 0\\0 & 0\end{bmatrix}+\begin{bmatrix}0.25 & 0\\0 & 0\end{bmatrix}+\begin{bmatrix}2.25 & 1.5\\1.5 & 1\end{bmatrix}+\begin{bmatrix}2.25 & 1.5\\1.5 & 1\end{bmatrix}\right)=\begin{bmatrix}1.25 & 0.75\\0.75 & 0.5\end{bmatrix}$$

$$i)\ \mathbf{C}_{zz}=\begin{bmatrix}1.875\\1.125\end{bmatrix}\cdot\begin{bmatrix}1.875 & 1.125\end{bmatrix}=\begin{bmatrix}3.516 & 2.109\\2.109 & 1.266\end{bmatrix}$$
$$ii)\ \mathbf{C}_{zz}=\begin{bmatrix}0.75\\0\end{bmatrix}\cdot\begin{bmatrix}0.75 & 0\end{bmatrix}=\begin{bmatrix}0.5625 & 0\\0 & 0\end{bmatrix}$$

$$\mathbf{C}_{mm}=\frac{1}{2}\cdot\left(c^2\cdot\begin{bmatrix}0.25 & 0\\0 & 0.25\end{bmatrix}+\begin{bmatrix}1.25 & 0.75\\0.75 & 0.5\end{bmatrix}\right)$$
$$\Rightarrow i)\ \mathbf{C}_{mm}=\begin{bmatrix}0.657 & 0.375\\0.375 & 0.281\end{bmatrix}\quad ii)\ \mathbf{C}_{mm}=\begin{bmatrix}1.125 & 0.375\\0.375 & 0.75\end{bmatrix}$$

$$\Rightarrow i)\ [\mathbf{C}_{mm}]^{-1}=\begin{bmatrix}6.388 & -8.524\\-8.524 & 14.935\end{bmatrix}\quad ii)\ [\mathbf{C}_{mm}]^{-1}=\begin{bmatrix}1.067 & -0.533\\-0.533 & 1.600\end{bmatrix}$$

Reliability criteria:

$$i)\ J_1=\mathrm{tr}\left\{\begin{bmatrix}6.388 & -8.524\\-8.524 & 14.935\end{bmatrix}\cdot\begin{bmatrix}3.516 & 2.109\\2.109 & 1.266\end{bmatrix}\right\}=\mathrm{tr}\left\{\begin{bmatrix}4.483 & 2.681\\1.527 & 0.931\end{bmatrix}\right\}=5.4$$

$$ii)\ J_1=\mathrm{tr}\left\{\begin{bmatrix}1.067 & -0.533\\-0.533 & 0.281\end{bmatrix}\cdot\begin{bmatrix}0.5625 & 0\\0 & 0\end{bmatrix}\right\}=\mathrm{tr}\left\{\begin{bmatrix}0.6 & 0\\-0.3 & 0\end{bmatrix}\right\}=0.6$$

The second case indicates clearly worse classification reliability.

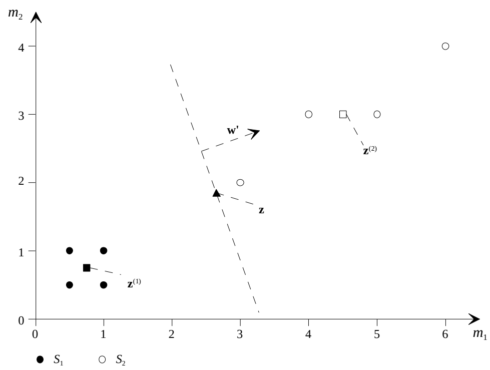
b) $\left[\mathbf{z}^{(1)}-\mathbf{z}^{(2)}\right]^{\mathrm{T}}\cdot\left[\mathbf{C}_{mm}\right]^{-1}\cdot\left[\mathbf{m}_0-\dfrac{\mathbf{z}^{(1)}+\mathbf{z}^{(2)}}{2}\right]=0$ as $P(S_1)=P(S_2)=0.5$

$$i)\ \begin{bmatrix}-3.75 & -2.25\end{bmatrix}\cdot\begin{bmatrix}6.388 & -8.524\\ -8.524 & 14.935\end{bmatrix}\cdot\left(\mathbf{m}_0-\begin{bmatrix}2.625\\1.875\end{bmatrix}\right)=\underbrace{\begin{bmatrix}-4.8 & -1.6\end{bmatrix}}_{\mathbf{w}^{\mathrm{T}}}\cdot\left(\mathbf{m}_0-\underbrace{\begin{bmatrix}2.625\\1.875\end{bmatrix}}_{\mathbf{z}}\right)=0$$

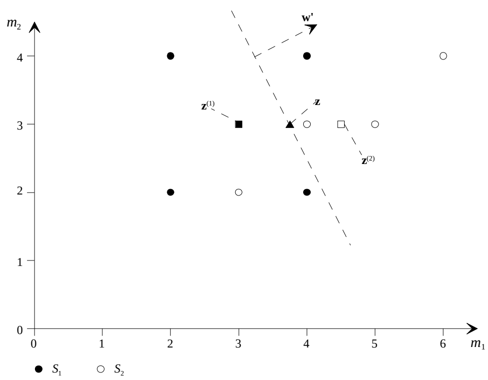
$$ii)\ \begin{bmatrix}-1.5 & 0\end{bmatrix}\cdot\begin{bmatrix}1.067 & -0.533\\ -0.533 & 1.600\end{bmatrix}\cdot\left(\mathbf{m}_0-\begin{bmatrix}3.75\\3\end{bmatrix}\right)=\underbrace{\begin{bmatrix}-1.6 & -0.8\end{bmatrix}}_{\mathbf{w}^{\mathrm{T}}}\cdot\left(\mathbf{m}_0-\underbrace{\begin{bmatrix}3.75\\3\end{bmatrix}}_{\mathbf{z}}\right)=0$$

The separation line crosses the point **z** and is perpendicular with **w**. In the following Figures, **w'** is a vector which points into the same direction as **w**.

c) c=0.5 : No wrong classification.

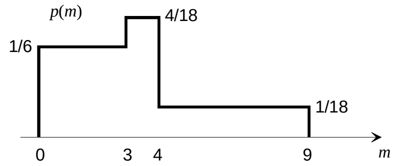


c=2 : $\mathbf{m}_4^{(1)}$ and $\mathbf{m}_4^{(2)}$ are wrongly classified.

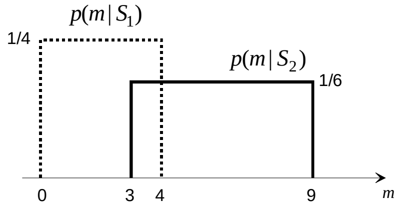


Problem 9.2

a) $z^{(1)} = (4+0)/2 = 2 \quad ; \quad z^{(2)} = (9+3)/2 = 6$

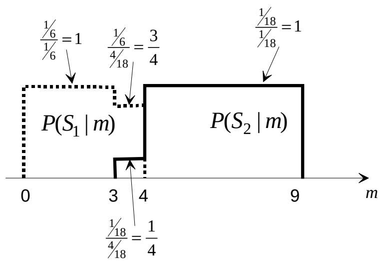


b) $P(S_1) = \frac{1}{6} \cdot 4 = \frac{2}{3} \quad ; \quad P(S_2) = \frac{1}{18} \cdot 6 = \frac{1}{3}$; a priori probabilities: see Figure below.



c) Bayes rule :

$$P(S_1 \mid m) = \frac{p(m \mid S_1) \cdot P(S_1)}{p(m)} \quad ; \quad P(S_2 \mid m) = \frac{p(m \mid S_2) \cdot P(S_2)}{p(m)}$$



d) Optimum threshold θ=4, as $P(S_1|m) > P(S_2|m)$ for $m < 4$. No wrong classification in S_1. Probability of wrong classification in S_2: $P(S_1|S_2)$=1/6, overall probability of wrong classification is $P_{\text{wrong}} = P(S_1|S_2) \cdot P(S_2)$=1/18.

e) Condition : Same areas in the range of overlap:

$$\tfrac{1}{18} \cdot (\theta_{=} - 3) = \tfrac{1}{6} \cdot (4 - \theta_{=}) \Rightarrow \theta_{=} = 3.75$$

$\Rightarrow P(S_2|S_1)$=1/16, $P(S_1|S_2)$=1/8, $P_{\text{wrong}} = P(S_2|S_1) \cdot P(S_1) + P(S_1|S_2) \cdot P(S_2)$=2/48+1/24=1/12

Problem 9.3

a) $\mathbf{z}^{(1)} = \begin{bmatrix} 2.5 \\ 2.5 \end{bmatrix} \quad ; \quad \mathbf{z}^{(2)} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} \quad ; \quad \mathbf{z}^{(3)} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$

Covariances are zero due to statistical independency of the features (joint probability is product of the first-order probabilities). Variances within all classes are identical for features m_1 and m_2:

$$\sigma^2_{m_1}{}^{(1)} = \sigma^2_{m_2}{}^{(1)} = \frac{3^2}{12} \quad ; \quad \sigma^2_{m_1}{}^{(2)} = \sigma^2_{m_2}{}^{(2)} = \frac{4^2}{12} \quad ; \quad \sigma^2_{m_1}{}^{(3)} = \sigma^2_{m_2}{}^{(3)} = \frac{2^2}{12}$$

$$\mathbf{C}_{mm}{}^{(1)} = \begin{bmatrix} \frac{3}{4} & 0 \\ 0 & \frac{3}{4} \end{bmatrix} \quad ; \quad \mathbf{C}_{mm}{}^{(2)} = \begin{bmatrix} \frac{4}{3} & 0 \\ 0 & \frac{4}{3} \end{bmatrix} \quad ; \quad \mathbf{C}_{mm}{}^{(3)} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

b)

$$p(\mathbf{m} \mid S_1) = \frac{1}{9} \quad \Rightarrow \quad p(\mathbf{m} \mid S_1)P(S_1) = \frac{1}{36}$$

$$p(\mathbf{m} \mid S_2) = \frac{1}{16} \quad \Rightarrow \quad p(\mathbf{m} \mid S_2)P(S_2) = \frac{3}{80}$$

$$p(\mathbf{m} \mid S_3) = \frac{1}{4} \quad \Rightarrow \quad p(\mathbf{m} \mid S_3)P(S_3) = \frac{3}{80}$$

c)

$$A: P(S_1 \mid \mathbf{m}) = \frac{p(\mathbf{m} \mid S_1)P(S_1)}{p(\mathbf{m})} = \frac{\frac{1}{36}}{\frac{1}{36}+\frac{3}{80}} = \frac{1}{1+\frac{27}{20}} = \frac{20}{47}$$

$$P(S_2 \mid \mathbf{m}) = \frac{p(\mathbf{m} \mid S_2)P(S_2)}{p(\mathbf{m})} = \frac{1}{1+\frac{20}{27}} = \frac{27}{47}$$

$$B: P(S_2 \mid \mathbf{m}) = \frac{p(\mathbf{m} \mid S_2)P(S_2)}{p(\mathbf{m})} = \frac{\frac{1}{36}}{\frac{1}{36}+\frac{1}{36}} = \frac{1}{2}$$

$$P(S_3 \mid \mathbf{m}) = \frac{p(\mathbf{m} \mid S_3)P(S_3)}{p(\mathbf{m})} = \frac{\frac{1}{36}}{\frac{1}{36}+\frac{1}{36}} = \frac{1}{2}$$

d) MAP classification: Within the area "A" $P(S_2|\mathbf{m})>P(S_1|\mathbf{m})$, hence all S_1 are wrongly classified here. Area "A" is 1/9 of the entire area of $S_1 \Rightarrow P(S_1|S_2)=0$, $P(S_2|S_1)=1/9$. Within the area "B" $P(S_2|\mathbf{m})=P(S_3|\mathbf{m})$: Half of all S_2 and S_3 within this area are wrongly classified $\Rightarrow P(S_2|S_3)=1/2\cdot 1/4=1/8$, $P(S_3|S_2)=1/2\cdot 1/16=1/32$

Total classification error:

$P_{\text{wrong}}= P(S_2|S_1)P(S_1) + P(S_3|S_2)P(S_2) + P(S_2|S_3)P(S_3)=1/64+3/160+3/160=17/320$

Problem 10.1

a) Θ=1,5: $\qquad$ Θ=3: $\qquad$ Θ=4,5 :

$$\mathbf{S} = \begin{bmatrix} 1 & 2 & 2 & 2 & 2 \\ 1 & 1 & 2 & 2 & 2 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 1 & 2 & 2 & 2 \\ 1 & 1 & 1 & 2 & 2 \end{bmatrix} \qquad \mathbf{S} = \begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ 1 & 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 & 2 \\ 1 & 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix} \qquad \mathbf{S} = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

b) Overall mean: $\mu = \frac{1}{25}\cdot\left[9\cdot 1+5\cdot 2+5\cdot 4+6\cdot 5\right] = 2.76$

Θ=1.5 :

μ_{S1}=1 ; σ^2_{S1}=0 ;

$$\mu_{S2} = \frac{1}{16} \cdot [5 \cdot 2 + 5 \cdot 4 + 6 \cdot 5] = 3.75 \,; \sigma^2{}_{S2} = \frac{1}{16} \cdot \left[5 \cdot 2^2 + 5 \cdot 4^2 + 6 \cdot 5^2\right] - 3{,}75^2 = 1.5625$$

$$P(S_1) = \frac{9}{25} = 0.36 \;;\; P(S_2) = \frac{16}{25} = 0.64$$

$$\Rightarrow \sigma_\mu{}^2 = 0.36 \cdot (1 - 2.76)^2 + 0.64 \cdot (3.75 - 2.76)^2 = 1.7424 \;;\; \bar{\sigma}^2 = 0.36 \cdot 0 + 0.64 \cdot 1.5625 = 1$$

$$\Rightarrow \frac{\sigma_\mu{}^2}{\bar{\sigma}^2} = 1.7424$$

Θ=3 :

$$\mu_{S1} = \frac{1}{14} \cdot [9 \cdot 1 + 5 \cdot 2] = 1.357 \,; \sigma^2{}_{S1} = \frac{1}{14} \cdot \left[9 \cdot 1^2 + 5 \cdot 2^2\right] - 1.357^2 = 0.23$$

$$\mu_{S2} = \frac{1}{11} \cdot [5 \cdot 4 + 6 \cdot 5] = 4.545 \,; \sigma^2{}_{S2} = \frac{1}{11} \cdot \left[5 \cdot 4^2 + 6 \cdot 5^2\right] - 4.545^2 = 0.248$$

$$P(S_1) = \frac{14}{25} = 0.56 \;;\; P(S_2) = \frac{11}{25} = 0.44$$

$$\Rightarrow \sigma_\mu{}^2 = 0.56 \cdot (1.375 - 2.76)^2 + 0.44 \cdot (4.545 - 2.76)^2 = 2.476 \;;$$

$$\bar{\sigma}^2 = 0.56 \cdot 0.23 + 0.44 \cdot 0.248 = 0.238 \Rightarrow \frac{\sigma_\mu{}^2}{\bar{\sigma}^2} = 10.40$$

Θ=4.5 :

$$\mu_{S1} = \frac{1}{19} \cdot [9 \cdot 1 + 5 \cdot 2 + 5 \cdot 4] = 2.053 \,; \sigma^2{}_{S1} = \frac{1}{19} \cdot \left[9 \cdot 1^2 + 5 \cdot 2^2 + 5 \cdot 4^2\right] - 2.053^2 = 1.522 \,;$$

μ_{S2}=5 ; σ^2_{S2}=0 ;

$$P(S_1) = \frac{19}{25} = 0.76 \;;\; P(S_2) = \frac{9}{25} = 0.24$$

$$\Rightarrow \sigma_\mu{}^2 = 0.76 \cdot (2.053 - 2.76)^2 + 0.24 \cdot (5 - 2.76)^2 = 1.584 \;;$$

$$\bar{\sigma}^2 = 0.76 \cdot 1.522 + 0.24 \cdot 0 = 1.157 \Rightarrow \frac{\sigma_\mu{}^2}{\bar{\sigma}^2} = 1.369$$

Optimum threshold segmentation is achieved for Θ=3.

c)

$$P(S_2 \,|\, S_1) = \frac{1}{\sqrt{2}\sigma_{S1}} \cdot \int_{\Theta}^{\infty} e^{-\frac{\sqrt{2}|x-\mu_{S1}|}{\sigma_{S1}}} dx = \frac{1}{\sqrt{2}\sigma_{S1}} \cdot \int_{\Theta-\mu_{S1}}^{\infty} e^{-\frac{\sqrt{2}x}{\sigma_{S1}}} dx$$

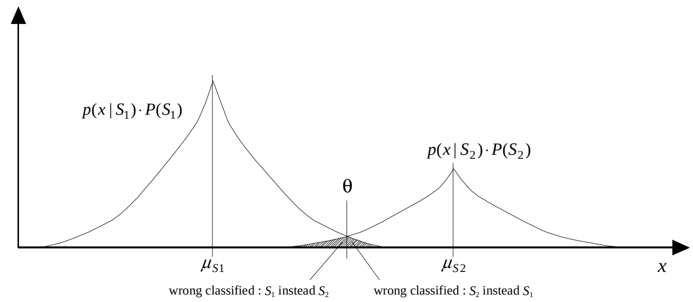
$$= \frac{1}{\sqrt{2}\sigma_{S1}} \cdot \left(-\frac{\sigma_{S1}}{\sqrt{2}}\right) \cdot \left| e^{-\frac{\sqrt{2}x}{\sigma_{S1}}} \right|_{\Theta-\mu_{S1}}^{\infty} = \frac{1}{2} \cdot e^{-\frac{\sqrt{2}\left(\Theta-\mu_{S1}\right)}{\sigma_{S1}}}$$

$$P(S_1\,|\,S_2)=\frac{1}{\sqrt{2}\sigma_{S2}}\cdot\int_{-\infty}^{\Theta}e^{-\frac{\sqrt{2}|x-\mu_{S2}|}{\sigma_{S2}}}dx=\frac{1}{\sqrt{2}\sigma_{S2}}\cdot\int_{\mu_{S2}-\Theta}^{\infty}e^{-\frac{\sqrt{2}x}{\sigma_{S2}}}dx$$

$$=\frac{1}{\sqrt{2}\sigma_{S2}}\cdot\left(-\frac{\sigma_{S2}}{\sqrt{2}}\right)\cdot\left|e^{-\frac{\sqrt{2}x}{\sigma_{S2}}}\right|_{\mu_{S2}-\Theta}^{\infty}=\frac{1}{2}\cdot e^{-\frac{\sqrt{2}(\mu_{S2}-\Theta)}{\sigma_{S2}}}$$

$P_{err}=P(S_2\,|\,S_1)\cdot P(S_1)+P(S_1\,|\,S_2)\cdot P(S_2)$;

$\Theta=3 \Rightarrow P_{err}=0.56\cdot 0.0039+0.44\cdot 0.0062=0.0049\approx 0.5$ %



Problem 10.2

a)

$$\mathbf{I}_{\mathrm{A}}=\begin{bmatrix}1&2&2&2&3\\1&2&2&2&3\\1&2&2&2&3\end{bmatrix}\quad;\quad\mathbf{I}_{\mathrm{B}}=\begin{bmatrix}1&1&2&3&3\\1&1&2&3&3\\1&1&2&3&3\end{bmatrix}$$

b) Θ_A: $P(S_1)=1/5; P(S_2)=3/5; P(S_3)=1/5$; $\mu_{S_1}=1; \mu_{S_2}=3; \mu_{S_3}=5$; $\sigma_{S_1}{}^2=0; \sigma_{S_2}{}^2=2/3; \sigma_{S_3}{}^2=0$
Θ_B: $P(S_1)=2/5; P(S_2)=1/5; P(S_3)=2/5$; $\mu_{S_1}=1{,}5; \mu_{S_2}=2; \mu_{S_3}=4{,}5$; $\sigma_{S_1}{}^2=1/4; \sigma_{S_2}{}^2=0; \sigma_{S_3}{}^2=1/4$

c) $\overline{\sigma^2}(\boldsymbol{\theta}_{\mathrm{A}})=\frac{1}{5}\cdot 0+\frac{3}{5}\cdot\frac{2}{3}+\frac{1}{5}\cdot 0=\frac{2}{5}$; $\sigma_\mu{}^2(\boldsymbol{\theta}_{\mathrm{B}})=\frac{1}{5}\cdot(1-3)^2+\frac{3}{5}\cdot(3-3)^2+\frac{1}{5}\cdot(5-3)^2=\frac{8}{5}$

$\overline{\sigma^2}(\boldsymbol{\theta}_B)=\frac{2}{5}\cdot\frac{1}{4}+\frac{1}{5}\cdot 0+\frac{2}{5}\cdot\frac{1}{4}=\frac{1}{5}$;

$\sigma_\mu{}^2(\boldsymbol{\theta}_{\mathrm{B}})=\frac{2}{5}\cdot(1{,}5-3)^2+\frac{1}{5}\cdot(3-3)^2+\frac{2}{5}\cdot(4{,}5-3)^2=\frac{9}{5}$

$\sigma_\mu{}^2(\boldsymbol{\theta}_{\mathrm{A}})/\overline{\sigma^2}(\boldsymbol{\theta}_{\mathrm{A}})=4$; $\sigma_\mu{}^2(\boldsymbol{\theta}_{\mathrm{A}})/\overline{\sigma^2}(\boldsymbol{\theta}_{\mathrm{A}})=9$, Θ_B is the better choice.

d) With $\Theta_C=[\,1\;\;5\,]^T$, $P(S_1)=0; P(S_2)=1; P(S_3)=0$; $\mu_{S_2}=3; \sigma_{S_2}{}^2=7/5$ $\Rightarrow \frac{\sigma_\mu{}^2(\boldsymbol{\theta}_{\mathrm{C}})}{\overline{\sigma^2}(\boldsymbol{\theta}_{\mathrm{C}})}=\frac{0}{7/5}=0$,

which is the lowest possible value.

Problem 11.1

a)

$$R(D)=\frac{1}{2}\log_2\frac{\sigma_z{}^2}{D}=\frac{1}{2}\left[\log_2\sigma_z{}^2-\log_2 D\right]$$

$$=\frac{1}{4\pi}\int_{-\pi}^{\pi}\log_2\frac{S_{xx}(\Omega)}{D}d\Omega=\frac{1}{2}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\log_2 S_{xx}(\Omega)d\Omega-\underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi}\log_2 D\,d\Omega}_{=\log_2 D}\right]$$

$$\Rightarrow\frac{1}{2\pi}\int_{-\pi}^{\pi}\log_2 S_{xx}(\Omega)d\Omega=\log_2\sigma_z{}^2$$

$$\gamma_x{}^2=\frac{2^{\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\log_2 S_{xx}(\Omega)d\Omega\right]}}{\sigma_x{}^2}=\frac{2^{\log_2\sigma_z{}^2}}{\sigma_x{}^2}=\frac{\sigma_z{}^2}{\sigma_x{}^2}$$

b)

$$R_G=\underbrace{\frac{1}{2}\log_2\frac{\sigma_x{}^2}{D}}_{R(D)\,uncorrelated}-\underbrace{\frac{1}{2}\log_2\frac{\sigma_z{}^2}{D}}_{R(D)\,correlated}=\frac{1}{2}\log_2\frac{\sigma_x{}^2}{D}-\frac{1}{2}\log_2\frac{\sigma_x{}^2\left(1-\rho^2\right)}{D}$$

$$=\frac{1}{2}\log_2\frac{\sigma_x{}^2}{D}-\frac{1}{2}\log_2\frac{\sigma_x{}^2}{D}-\frac{1}{2}\log_2\left(1-\rho^2\right)=-\frac{1}{2}\log_2\left(1-\rho^2\right)$$

$$\text{2D}:\sigma_z{}^2=\sigma_x{}^2\cdot\left(1-\rho_h{}^2\right)\cdot\left(1-\rho_v{}^2\right)$$

$$\Rightarrow R_G=\frac{1}{2}\log_2\frac{\sigma_x{}^2}{D}-\frac{1}{2}\log_2\frac{\sigma_x{}^2\cdot\left(1-\rho_h{}^2\right)\cdot\left(1-\rho_v{}^2\right)}{D}=-\frac{1}{2}\log_2\left(1-\rho_h{}^2\right)-\frac{1}{2}\log_2\left(1-\rho_v{}^2\right)$$

$\rho_h=\rho_v=0.95$: $R_{G,1\text{D}}=1.679$ bit ; $R_{G,2\text{D}}=1.679+1.679=3.358$ bit

c)

$$R(D_{\max})=\frac{1}{2}\log_2\frac{\sigma_x{}^2\cdot\left(1-\rho_h{}^2\right)\cdot\left(1-\rho_v{}^2\right)}{D_{\max}}$$

$$D_{\max}\le S_{xx}(\pi,\pi)=\sigma_x{}^2\cdot\frac{(1-\rho_h{}^2)}{1+2\rho_h+\rho_h{}^2}\cdot\frac{(1-\rho_v{}^2)}{1+2\rho_v+\rho_v{}^2}$$

$$=\sigma_x{}^2\cdot\frac{(1-\rho_h)\cdot(1+\rho_h)}{(1+\rho_h)^2}\cdot\frac{(1-\rho_v)\cdot(1+\rho_v)}{(1+\rho_v)^2}$$

$$=\sigma_x{}^2\cdot\frac{(1-\rho_h)}{(1+\rho_h)}\cdot\frac{(1-\rho_v)}{(1+\rho_v)}$$

$$\Rightarrow R(D_{\max})=\frac{1}{2}\log_2\frac{\sigma_x{}^2\cdot(1-\rho_h{}^2)\cdot(1-\rho_v{}^2)}{\sigma_x{}^2\cdot\frac{(1-\rho_h)}{(1+\rho_h)}\cdot\frac{(1-\rho_v)}{(1+\rho_v)}}$$

$$=\frac{1}{2}\log_2\left[(1+\rho_h)^2\cdot(1+\rho_h)^2\right]=\log_2\left[(1+\rho_h)\cdot(1+\rho_h)\right]$$

$\rho_h=\rho_v=0.95$: $R(D_{\max})=1.92$ bit

d)

$$S_{xx}(\Omega)=\frac{\sigma_x{}^2(1-\rho^2)}{1-2\rho\cos\Omega+\rho^2}$$

$$\Theta(\Omega_{\max} = \pi/4) = S_{xx}(\pi/4) = \frac{\sigma_x{}^2(1-\rho^2)}{1-2\rho\cdot\frac{\sqrt{2}}{2}+\rho^2} = \sigma_x{}^2\cdot\frac{1-\rho^2}{1-\sqrt{2}\rho+\rho^2}$$

$$\Theta(\Omega_{\max} = \pi/2) = S_{xx}(\pi/2) = \frac{\sigma_x{}^2(1-\rho^2)}{1-2\rho\cdot 0+\rho^2} = \sigma_x{}^2\cdot\frac{1-\rho^2}{1+\rho^2}$$

$$\Theta(\Omega_{\max} = \pi) = S_{xx}(\pi) = \frac{\sigma_x{}^2(1-\rho^2)}{1+2\rho+\rho^2} = \sigma_x{}^2\cdot\frac{1-\rho}{1+\rho}$$

Θ/σ_x^2	$\Omega_{max}=\pi/4$	$\Omega_{max}=\pi/2$	$\Omega_{max}=\pi$
$\rho=0.5$	1.3815	0.6	0.3333
$\rho=0.95$	0.1744	0.0512	0.0256

Problem 11.2
Entropy: 1.92 bit
Shannon code: z_1=2 bit, z_2= z_3= z_4=3 bit, R=2.6 bit
Huffman code: z_1=1 bit, z_2=2 bit, z_3= z_4=3 bit or z_1=z_2=z_3= z_4=2 bit, R=2 bit in both cases.

Problem 11.3

a) $H(j) = -[0{,}25\cdot\log_2 0{,}25 + 0{,}75\cdot\log_2 0{,}75] = 0{,}5 + 0{,}311 = 0{,}811\,bit$

b) $P(0,0) = 0{,}25\cdot 0{,}25 = 0{,}0625 = 2^{-4}$

$P(0,1) = P(1,0) = 0{,}25\cdot 0{,}75 = 0{,}1875 = 3\cdot 2^{-4}$

$P(1,1) = 0{,}75\cdot 0{,}75 = 0{,}5625 = 9\cdot 2^{-4}$

$P(0,0,0) = 0{,}25^3 = 0{,}015625 = 2^{-6}$

$P(0,0,1) = P(0,1,0) = P(1,0,0) = 0{,}25^2\cdot 0{,}75 = 0{,}046875 = 3\cdot 2^{-6}$

$P(0,1,1) = P(1,1,0) = P(1,0,1) = 0{,}75^2\cdot 0{,}25 = 0{,}140625 = 9\cdot 2^{-6}$

$P(1,1,1) = 0{,}75^3 = 0{,}421875 = 27\cdot 2^{-6}$

c) $i(j) = -\log_2 P(j)$; $z(j)$ allocated number of bits

First-order probability : Encoding of separate source symbols

source symbol	$i(j)$ [bit]	$z(j)_{\text{Shannon}}$	$z(j)_{\text{Huffman}}$
0	2	2	1
1	0.415	1	1

R_{Shannon}=0.25·2 bit+0.75·1 bit=1.25 bit
R_{Huffman}=0.25·1 bit+0.75·1 bit=1 bit

Joint probability of second order: Encoding of source symbol vectors, K=2

source symbol	$i(j)$ [bit]	$z(j)_{\text{Shannon}}$	$z(j)_{\text{Huffman}}$
0,0	4	4	3
0,1	2.415	3	3
1,0	2.415	3	2
1,1	0.830	1	1

R_{Shannon}=(0.0625·4 bit+2·0.1875·3 bit+0.5625·1 bit):2 =0.96875 bit = H + 0.1578 bit
R_{Huffman}=(0.0625·3 bit+0.1875·3 bit+0.1875·2 bit +0.5625·1 bit):2=0.84375 bit
= H + 0.03275 bit

Joint probability of third order: Encoding of source symbol vectors, K=3:

source symbol	$i(j)$ [bit]	$z(j)_{\text{Shannon}}$	$z(j)_{\text{Huffman}}$
0,0,0	6	6	5
0,0,1	4.415	5	5
0,1,0	4.415	5	5
1,0,0	4.415	5	5
0,1,1	2.830	3	3
1,0,1	2.830	3	3
1,1,0	2.830	3	3
1,1,1	1.245	2	1

R_{Shannon}=(0.015625·6 bit+3·0.046875·5 bit+3·0.140625·3 bit+0.421875·2 bit):3
=0.96875 bit = H + 0.1578 bit
R_{Huffman}=(0.015625·5 bit+3·0.046875·5 bit+3·0.140625·3 bit+0.421875·1 bit):3
=0.8229 bit = H + 0.0119 bit

d) Using (3.77) :

$$P(0\,|\,1) = P(0)\cdot\left[P(0\,|\,1)+P(1\,|\,0)\right] \quad \Rightarrow P(0\,|\,1)\cdot\left[1-P(0)\right] = P(0)\cdot P(1\,|\,0)$$

$$P(0\,|\,1) = \frac{P(0)\cdot P(1\,|\,0)}{P(1)} = \frac{0.25\cdot 0.5}{0.75} = 0.166\overline{6}$$

$$P(0\,|\,0) = 1-P(1\,|\,0) = 0.5\ ;\ \ P(1\,|\,1) = 1-P(0\,|\,1) = 0.833\overline{3}$$

e) Using (3.26) :

$$P(0,0) = P(0\,|\,0)\cdot P(0) = 0.5\cdot 0.25 = 0.125$$

$$P(0,1) = P(0\,|\,1)\cdot P(1) = 0.166\overline{6}\cdot 0.75 = 0.125$$

$$P(1,0) = P(1\,|\,0)\cdot P(0) = 0.5\cdot 0.25 = 0.125$$

$$P(1,1) = P(1\,|\,1)\cdot P(1) = 0.833\overline{3}\cdot 0.75 = 0.625$$

The bit allocation of the Huffman code is identical with the result of c). As however the probabilities are different, the rate is:
R_{Huffman}=(2·0.125·3 bit+0.125·2 bit +0.625·1 bit):2=0.8125 bit

f) Using (11.47)-(11.49) :

$$H(b)\big|_{b=1} = 0.166\overline{6}\cdot 2.58496 + 0.833\overline{3}\cdot 0.26303 = 0.650022 \text{ bit}$$

$$H(b)\big|_{b=0} = 0.5\cdot 1 + 0.5\cdot 1 = 1 \text{ bit}$$

$$H(b) = 0.25\cdot 1 + 0.75\cdot 0.650022 = 0.73751 \text{ bit}$$

Problem 11.4

a) $H(j) = 0.8\cdot\log_2 0.8 + 0.2\cdot\log_2 0.2 = 0.7219$ bit

b) Boundaries of probability intervals:
0["A"]0.8["B"]1
0["AA"]0.64["AB"]0.8["BA"]0.96["BB"]1
0["AAA"]0.512["AAB"]0.64["ABA"]0.768["ABB"]0.8["BAA"]0.928["BAB"]0.96["BBA"]0.992["BBB"]1

c) Boundaries of code intervals, 3 bit accuracy :
0 ["0"] 0.5 ["1"] 1
0 ["00"] 0.25 ["01"] 0.5 ["10"] 0.75 ["11"] 1
0 ["000"] 0.125 ["001"] 0.25 ["010"] 0.375 ["011"] 0.5 ["100"] 0.625 ["101"] 0.75 ["110"] 0.875 ["111"] 1

Boundaries of probability intervals, rounded; underlined intervals exactly correspond with code intervals underlined above, which results in the following mapping:

0["A"]0.75<u>["B"]</u>1 "B"→"11"
0["AA"]0.625<u>["AB"]</u>0.75 "AB"→"101"
0<u>["AAA"]</u>0.5<u>["AAB"]</u>0.625 "AAA"→"0" ; "AAB"→"100"

The computation of the mean rate (per source symbol) must be modified as compared to (11.29), since the number of source symbols K_j must be considered individually for each code symbol j:

$$R=\sum_j P(j)\cdot z_j\cdot\frac{1}{K_j}$$

$$P("B")=0.2\ ;\ P("AB")=0.8\cdot 0.2=0.16$$

$$P("AAA")=0.8^3=0.512\ ;\ P("AAB")=0.8^2\cdot 0.2=0.128$$

$$\Rightarrow R=\underbrace{0.2\cdot 2\cdot\frac{1}{1}}_{"B"}+\underbrace{0.16\cdot 3\cdot\frac{1}{2}}_{"AB"}+\underbrace{0.512\cdot 1\cdot\frac{1}{3}}_{"AAA"}+\underbrace{0.128\cdot 3\cdot\frac{1}{3}}_{"AAB"}$$

$$=0.4+0.24+0.1706\overline{6}+0.128=0.9386\overline{6}\ \ bit$$

Problem 11.5

a) Method 1 (Fig. 11.18a): 1-2-4-1-1-1-1-2-3. Method 2 (Fig. 11.18b): 1-0-4-1-1-0-3:

b) $P_0(l)=P(1|0)\cdot\left[P(0|0)\right]^{l-1}\ ;\ P_1(l)=P(0|1)\cdot\left[P(1|1)\right]^{l-1}$

$$P(1)=\frac{P(1|0)}{P(0|1)+P(1|0)}=0.333\overline{3}\ ;\ P(0)=\frac{P(0|1)}{P(0|1)+P(1|0)}=0.666\overline{6}$$

$$P_0(1)=P(1|0)=0.4$$

$$P_0(2)=P(1|0)\cdot P(0|0)=0,4\cdot 0,6=0.24$$

$$P_0(3)=P(1|0)\cdot P(0|0)^2=0,4\cdot 0,36=0.144$$

$$P_1(1)=P(0|1)=0.8$$

$$P_1(2)=P(0|1)\cdot P(1|1)=0.8\cdot 0.2=0.16$$

$$P_1(3)=P(0|1)\cdot P(1|1)^2=0.8\cdot 0.04=0.032$$

c) According to Fig. 11.18b, only the run-lengths $P_0(l)$ are encoded, but the special case l=0 must also be regarded. To get a consistent result, it is assumed that any run of "0"-bits starts after a "1"-bit and is terminated by reaching the state "1" again. This gives a probability for a run-length $l>0$

$$P_0(l)=P(0|1)\cdot P(0|0)^{l-1}\cdot P(1|0).$$

The probability of a run l=0 is simply

$$P_0(l=0)=P(1|1).$$

It is straightforward to show that all run-length probabilities sum up as unity:

$$P(1|1)+\sum_{l=1}^{\infty}P(0|1)\cdot P(0|0)^{l-1}\cdot P(1|0)=P(1|1)+P(0|1)\cdot P(1|0)\cdot\sum_{l=0}^{\infty}P(0|0)^{l}$$
$$=P(1|1)+P(0|1)\cdot P(1|0)\cdot\frac{1}{1-P(0|0)}=P(1|1)+P(0|1)\cdot P(1|0)\cdot\frac{1}{P(1|0)}$$
$$=P(1|1)+P(0|1)=1.$$

As all run-lengths above l=3 shall be handled by the ESCAPE symbol, the numerical results are as follows:

$P('l=0')=P(1|1)=0.2$

$P('l=1')=P(0|1)\cdot P(1|0)=0.8\cdot 0.4=0.32$

$P('l=2')=P(l=1)\cdot P(0|0)=0.32\cdot 0.6=0.192$

$P('l=3')=P(l=2)\cdot P(0|0)=0.192\cdot 0.6=0.1152$

$P('ESC')=1-0.2-0.32-0.192-0.1152=0.1728.$

Huffman code : ESCAPE and l=3 : 3 bits ; l=0, l=1 and l=2: 2 bits.

Problem 11.6

a) Training sequence vectors:

$\mathbf{x}(0)=[-1\ 0]^T$; $\mathbf{x}(1)=[-3\ -2]^T$; $\mathbf{x}(2)=[1\ 1]^T$; $\mathbf{x}(3)=[-5\ -4]^T$; $\mathbf{x}(4)=[0\ 1]^T$;
$\mathbf{x}(5)=[1\ 0]^T$; $\mathbf{x}(6)=[2\ 2]^T$

1st iteration: Squared distances according to (11.50)

	$\mathbf{x}(0)$	$\mathbf{x}(1)$	$\mathbf{x}(2)$	$\mathbf{x}(3)$	$\mathbf{x}(4)$	$\mathbf{x}(5)$	$\mathbf{x}(6)$
$\mathbf{y}_0$	$\underline{1}$	$\underline{5}$	8	$\underline{25}$	5	5	18
$\mathbf{y}_1$	5	25	$\underline{0}$	61	$\underline{1}$	$\underline{1}$	$\underline{2}$

Allocation for $\mathbf{y}_0$: $\mathbf{x}(0)$; $\mathbf{x}(1)$; $\mathbf{x}(3)$
Allocation for $\mathbf{y}_1$: $\mathbf{x}(2)$; $\mathbf{x}(4)$; $\mathbf{x}(5)$; $\mathbf{x}(6)$

$$\mathbf{y}^{(1)}{}_{0,opt}=\left[\frac{-1-3-5}{3}\quad\frac{0-2-4}{3}\right]^T=[-3\quad -2]^T$$
$$\mathbf{y}^{(1)}{}_{1,opt}=\left[\frac{1+0+1+2}{4}\quad\frac{1+1+0+2}{4}\right]^T=[1\quad 1]^T$$

2nd iteration: Squared distances according to (11.50)

	$\mathbf{x}(0)$	$\mathbf{x}(1)$	$\mathbf{x}(2)$	$\mathbf{x}(3)$	$\mathbf{x}(4)$	$\mathbf{x}(5)$	$\mathbf{x}(6)$
$\mathbf{y}^{(1)}{}_{0,\mathrm{opt}}$	8	$\underline{0}$	25	$\underline{8}$	18	20	41
$\mathbf{y}^{(1)}{}_{1,\mathrm{opt}}$	$\underline{5}$	25	$\underline{0}$	61	$\underline{1}$	$\underline{1}$	$\underline{2}$

Allocation for $\mathbf{y}_0$: $\mathbf{x}(1)$; $\mathbf{x}(3)$
Allocation for $\mathbf{y}_1$: $\mathbf{x}(0)$; $\mathbf{x}(2)$; $\mathbf{x}(4)$; $\mathbf{x}(5)$; $\mathbf{x}(6)$

$$\mathbf{y}^{(2)}{}_{0,opt}=\left[\frac{-3-5}{2}\quad\frac{-2-4}{2}\right]^T=[-4\quad -3]^T$$
$$\mathbf{y}^{(2)}{}_{1,opt}=\left[\frac{-1+1+0+1+2}{5}\quad\frac{0+1+1+0+2}{5}\right]^T=[0.6\quad 0.8]^T$$

b) After 1st iteration: $P(0)=3/7 \Rightarrow i(0)=1.2224$; $P(1)=4/7 \Rightarrow i(1)=0.8074$
2nd iteration: Distances according to cost function (11.67), λ=0 identical to result a)

2nd iteration: Distances according to cost function (11.67), λ=65 in table below

	x(0)	**x**(1)	**x**(2)	**x**(3)	**x**(4)	**x**(5)	**x**(6)
$\mathbf{y}^{(1)}{}_{0,\mathrm{opt}}$	8+79.5 =87.5	0+79.5 =79.5	25+79.5 =104.5	8+79.5 =87.5	18+79.5 =97.5	20+79.5 =99.5	41+79.5 =120.5
$\mathbf{y}^{(1)}{}_{1,\mathrm{opt}}$	5+52.5 =57.5	25+52.5 =77.5	0+52.5 =52.5	61+52.5 =113.5	1+52.5= 53.5	1+52.5= 53.5	2+52.5 =54.5

Allocation for $\mathbf{y}_0$: **x**(3)
Allocation for $\mathbf{y}_1$: **x**(0) ; **x**(1) ; **x**(2) ; **x**(4) ; **x**(5) ; **x**(6)

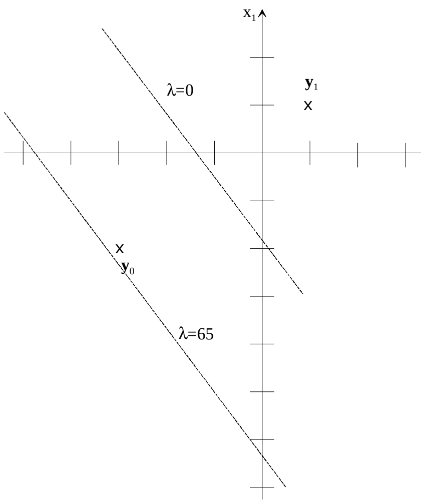
$$\mathbf{y}^{(2)}{}_{0,opt} = \begin{bmatrix} \frac{-5}{1} & \frac{-4}{1} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} -5 & -4 \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{y}^{(2)}{}_{1,opt} = \begin{bmatrix} \frac{-1-3+1+0+1+2}{6} & \frac{0-2+1+1+0+2}{5} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 0 & 0.4 \end{bmatrix}^{\mathrm{T}}$$

Equation of separation line according to (11.69):

$$2\begin{bmatrix} x_0+1 & x_1+0.5 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 3 \end{bmatrix} = -\lambda \cdot 0.415 \Rightarrow 8x_0+8+6x_1+3 = -\lambda \cdot 0.415$$

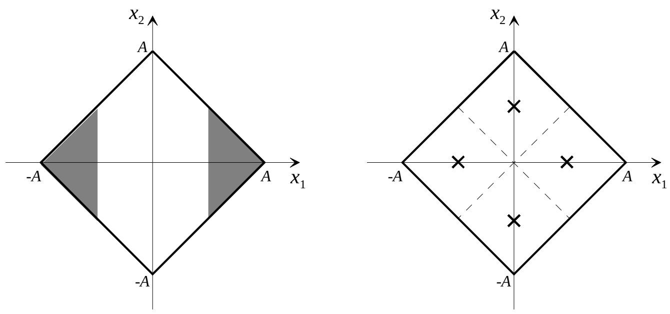
λ=0 : $\dfrac{x_0}{\left(-\frac{11}{8}\right)} + \dfrac{x_1}{\left(-\frac{11}{6}\right)} = 1$ λ=65 : $\lambda\cdot$0.415≈27 : $\dfrac{x_0}{\left(-\frac{19}{4}\right)} + \dfrac{x_1}{\left(-\frac{19}{3}\right)} = 1$



Problem 11.7

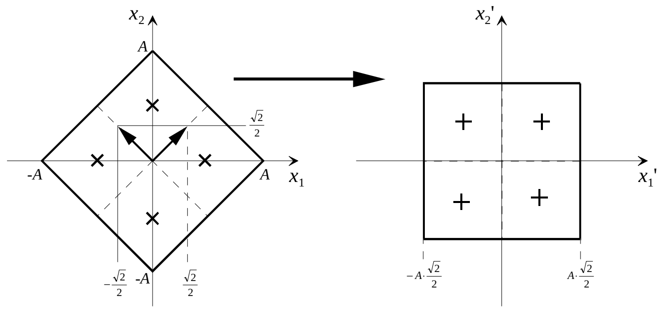
a) $$p(x_1,x_2) = \begin{cases} \frac{1}{2A^2}, |x_1|+|x_2| \le A \\ 0, \text{ else} \end{cases}$$

b) Solution graphically from triangular areas $P(|x_1| \geq A/2) = 2 \cdot \frac{1}{2A^2} \cdot \frac{1}{2} \cdot A \cdot \frac{A}{2} = 1/4$



c) Voronoi lines are principal axes of 45°, as all reconstruction values on coordinate axes have same distances from the origin. Optimum is a=A/2 (centroid of Voronoi regions at the centers of the squares, due to uniform PDF)

d) $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ (see Figure)



e) Quantization error identical within all Voronoi regions, can be derived by the $\Delta^2/12$ formula for uniform quantization error variance (related to samples; variance per vector would have double value) :

$$\sigma_q{}^2 = \frac{\left(\sqrt{2} \cdot A/2\right)^2}{12} = \frac{A^2}{24}$$

Problem 12.1

a) Image 1:

$$\frac{1}{MN} \sum_m \sum_n |x(m,n) - y(m,n)| = \frac{1}{9}\left[|20-19| + |17-18| + \ldots + |14-13|\right] = 1$$

Image 2 :

$$\frac{1}{MN}\sum_m\sum_n\left|x(m,n)-y(m,n)\right| = \frac{1}{9}\left[\left|20-20\right|+\left|17-17\right|+\ldots+\left|14-23\right|+\ldots+\left|14-14\right|\right]=1$$

b) Image 1:

$$\frac{1}{MN}\sum_m\sum_n\left(x(m,n)-y(m,n)\right)^2 = \frac{1}{9}\left[(20-19)^2+(17-18)^2+\ldots+(14-13)^2\right]=1$$

$$PSNR = 10\cdot\log_{10}\frac{255^2}{1} = 48.13\ \ dB$$

Image 2 :

$$\frac{1}{MN}\sum_m\sum_n\left|x(m,n)-y(m,n)\right| = \frac{1}{9}\left[(20-20)^2+(17-17)^2+\ldots+(14-23)^2+\ldots+(14-14)^2\right]=9$$

$$PSNR = 10\cdot\log_{10}\frac{255^2}{9} = 38.59\ \ dB$$

Interpretation: If the squared error (energy of error) criterion is used, single high deviations are penalized more strictly than for the case of errors which are widely dispersed over the image. This phenomenon is in harmony with the visibility of errors (single high errors are clearly visible, not being masked by the image content).

Problem 12.2

a)

i) $\hat{\mathbf{X}} = \begin{bmatrix} 0&0&0&0&0\\ 0&0&1&1&1\\ 0&0&1&1&1\\ 0&0&1&1&1\\ 0&0&0&0&0 \end{bmatrix}$ ii) $\hat{\mathbf{X}} = \begin{bmatrix} 0&0&0&0&0\\ 0&0&0.5&0.5&0.5\\ 0&0.5&1&1&0.5\\ 0&0.5&1&1&0.5\\ 0&0.5&0.5&0.5&0 \end{bmatrix}$ iii) $\hat{\mathbf{X}} = \begin{bmatrix} 0&0&0&0&0\\ 0&0&1&1&1\\ 0&1&1&1&0\\ 0&1&1&1&0\\ 0&1&0&0&-1 \end{bmatrix}$

b)

i) $\mathbf{E} = \begin{bmatrix} 0&0&0&0&0\\ 0&1&0&0&-1\\ 0&1&0&0&-1\\ 0&1&0&0&-1\\ 0&0&0&0&0 \end{bmatrix}$ ii) $\mathbf{E} = \begin{bmatrix} 0&0&0&0&0\\ 0&1&0.5&0.5&-0.5\\ 0&0.5&0&0&-0.5\\ 0&0.5&0&0&-0.5\\ 0&-0.5&-0.5&-0.5&0 \end{bmatrix}$ iii) $\mathbf{E} = \begin{bmatrix} 0&0&0&0&0\\ 0&1&0&0&-1\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&-1&0&0&1 \end{bmatrix}$

c)

i) $$\mathbf{Y} = \underbrace{\begin{bmatrix} 0&0&0&0&0\\ 0&0&0&0&-1\\ 0&1&0&0&-1\\ 0&1&0&0&-1\\ 0&0&0&0&0 \end{bmatrix}}_{\mathbf{E}'} + \underbrace{\begin{bmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&1&1&1\\ 0&0&1&1&1\\ 0&0&0&0&0 \end{bmatrix}}_{\hat{\mathbf{X}}'} = \begin{bmatrix} 0&0&0&0&0\\ 0&0&0&0&-1\\ 0&1&1&1&0\\ 0&1&1&1&0\\ 0&0&0&0&0 \end{bmatrix} \Rightarrow \mathbf{X}-\mathbf{Y} = \begin{bmatrix} 0&0&0&0&0\\ 0&1&1&1&1\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{bmatrix}$$

Impulse response of the predictor filter: z^{-1} , i.e. $\hat{\mathbf{X}}$ is generated step by step from **Y** by a shift of one sample. Feedback of the wrong prediction resulting by one transmission error – the deviation is the convolution of the transmission error by the impulse response of the inverse prediction error filter (synthesis filter), $B(z)=1/(1-z^{-1}) \Rightarrow b(n)$ is a discrete unit step function horizontally. Subsequently, the matrices for cases of both 2D predictors are given, which allow to better understand the effect of 2D error propagation.

ii)

$$\mathbf{Y} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & -0.5 \\ 0 & 0.5 & 0 & 0 & -0.5 \\ 0 & 0.5 & 0 & 0 & -0.5 \\ 0 & -0.5 & -0.5 & -0.5 & 0 \end{bmatrix}}_{\mathbf{E}'} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.25 & 0.375 \\ 0 & 0 & 0.5 & 0.625 & 0.25 \\ 0 & 0.25 & 0.625 & 0.625 & 0.375 \\ 0 & 0.375 & 0.25 & 0.375 & -0.125 \end{bmatrix}}_{\hat{\mathbf{X}}'}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0.75 & -0.125 \\ 0 & 0.5 & 0.5 & 0.625 & -0.25 \\ 0 & 0.75 & 0.625 & 0.625 & -0.125 \\ 0 & -0.125 & -0.25 & -0.125 & -0.125 \end{bmatrix} \Rightarrow \mathbf{X} - \mathbf{Y} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0.5 & 0.25 & 0.125 \\ 0 & 0.5 & 0.5 & 0.375 & 0.25 \\ 0 & 0.25 & 0.375 & 0.375 & 0.125 \\ 0 & 0.125 & 0.25 & 0.125 & 0.125 \end{bmatrix}$$

iii)

$$\mathbf{Y} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & -1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & -1 & -1 & -1 \end{bmatrix} \Rightarrow \quad \mathbf{X} - \mathbf{Y} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

d)

$$\mathbf{E} = \mathbf{X} - \hat{\mathbf{X}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2/3 & 1/3 & -1 \\ 0 & 1 & 2/3 & 1/3 & -1 \\ 0 & 1 & 2/3 & 1/3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{Y} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1/3 & 1/3 & 1/3 & -1/3 \\ 0 & 1/3 & 1/3 & 1/3 & -1/3 \\ 0 & 1/3 & 1/3 & 1/3 & -1/3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{V}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 2/3 & 1 \\ 0 & 0 & 1/3 & 2/3 & 1 \\ 0 & 0 & 1/3 & 2/3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\hat{\mathbf{X}}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1/3 & 2/3 & 1 & 2/3 \\ 0 & 1/3 & 2/3 & 1 & 2/3 \\ 0 & 1/3 & 2/3 & 1 & 2/3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \mathbf{Q} = \mathbf{X} - \mathbf{Y} = \mathbf{E} - \mathbf{V} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2/3 & 1/3 & 0 & -2/3 \\ 0 & 2/3 & 1/3 & 0 & -2/3 \\ 0 & 2/3 & 1/3 & 0 & -2/3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Interpretation: Feedback of quantization error. The values of **Q** appear shifted by one sample (convolution by impulse response of predicto filter: z^{-1}) in the prediction error signal **E**.

Problem 12.3

With U=2 according to (4.157):

$$\underbrace{\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}}_{\mathbf{T}} \cdot \underbrace{\sigma_x^2 \cdot \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}}_{\mathbf{R}_{xx}} \cdot \underbrace{\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}}_{\mathbf{T}^{\mathrm{T}}} = \frac{1}{2} \cdot \sigma_x^2 \begin{bmatrix} 1+\rho & 1+\rho \\ 1-\rho & \rho-1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \underbrace{\sigma_x^2 \cdot \begin{bmatrix} 1+\rho & 0 \\ 0 & 1-\rho \end{bmatrix}}_{\mathbf{\Lambda}}$$

$$E\left\{c_0^2\right\} = \sigma_x^2 \cdot (1+\rho) \quad ; \quad E\left\{c_1^2\right\} = \sigma_x^2 \cdot (1-\rho)$$

With U=3, using results from Problem 4.7:

$$E\left\{c_0^2\right\} = \sigma_x^2 \cdot \left(1 + \frac{4}{3}\rho + \frac{2}{3}\rho^2\right) \quad ; \quad E\left\{c_1^2\right\} = \sigma_x^2 \cdot \left(1-\rho^2\right) \quad ; \quad E\left\{c_2^2\right\} = \sigma_x^2 \cdot \left(1 - \frac{4}{3}\rho + \frac{1}{3}\rho^2\right)$$

U=V=2 :

$$E\left\{c_{00}{}^2\right\} = \sigma_x{}^2 \cdot (1+\rho_h)\cdot(1+\rho_v) \quad ; \quad E\left\{c_{01}{}^2\right\} = \sigma_x{}^2 \cdot (1+\rho_h)\cdot(1-\rho_v)$$
$$E\left\{c_{10}{}^2\right\} = \sigma_x{}^2 \cdot (1-\rho_h)\cdot(1+\rho_v) \quad ; \quad E\left\{c_{11}{}^2\right\} = \sigma_x{}^2 \cdot (1-\rho_h)\cdot(1-\rho_v)$$

U=V=3 :

$$E\left\{c_{00}{}^2\right\} = \sigma_x{}^2 \cdot \left(1+\frac{4}{3}\rho_h+\frac{2}{3}\rho_h{}^2\right)\cdot\left(1+\frac{4}{3}\rho_v+\frac{2}{3}\rho_v{}^2\right)$$
$$E\left\{c_{01}{}^2\right\} = \sigma_x{}^2 \cdot \left(1+\frac{4}{3}\rho_h+\frac{2}{3}\rho_h{}^2\right)\cdot\left(1-\rho_v{}^2\right)$$
$$E\left\{c_{02}{}^2\right\} = \sigma_x{}^2 \cdot \left(1+\frac{4}{3}\rho_h+\frac{2}{3}\rho_h{}^2\right)\cdot\left(1-\frac{4}{3}\rho_v+\frac{1}{3}\rho_v{}^2\right)$$
$$E\left\{c_{10}{}^2\right\} = \sigma_x{}^2 \cdot \left(1-\rho_h{}^2\right)\cdot\left(1+\frac{4}{3}\rho_v+\frac{2}{3}\rho_v{}^2\right)$$
$$E\left\{c_{11}{}^2\right\} = \sigma_x{}^2 \cdot \left(1-\rho_h{}^2\right)\cdot\left(1-\rho_v{}^2\right)$$
$$E\left\{c_{12}{}^2\right\} = \sigma_x{}^2 \cdot \left(1-\rho_h{}^2\right)\cdot\left(1-\frac{4}{3}\rho_v+\frac{1}{3}\rho_v{}^2\right)$$
$$E\left\{c_{20}{}^2\right\} = \sigma_x{}^2 \cdot \left(1-\frac{4}{3}\rho_h+\frac{1}{3}\rho_h{}^2\right)\cdot\left(1+\frac{4}{3}\rho_v+\frac{2}{3}\rho_v{}^2\right)$$
$$E\left\{c_{21}{}^2\right\} = \sigma_x{}^2 \cdot \left(1-\frac{4}{3}\rho_h+\frac{1}{3}\rho_h{}^2\right)\cdot\left(1-\rho_v{}^2\right)$$
$$E\left\{c_{22}{}^2\right\} = \sigma_x{}^2 \cdot \left(1-\frac{4}{3}\rho_h+\frac{1}{3}\rho_h{}^2\right)\cdot\left(1-\frac{4}{3}\rho_v+\frac{1}{3}\rho_v{}^2\right)$$

Coding gain of discrete transform (12.33)

$$\Rightarrow G_{\mathrm{TC,1D}} = \frac{\frac{1}{U}\sum_{u=0}^{U-1}E\left\{c_u{}^2\right\}}{\left[\prod_{u=0}^{U-1}E\left\{c_u{}^2\right\}\right]^{1/U}} \qquad G_{\mathrm{TC,2D}} = \frac{\frac{1}{U\cdot V}\sum_{u=0}^{U-1}\sum_{v=0}^{V-1}E\left\{c_{u,v}{}^2\right\}}{\left[\prod_{u=0}^{U-1}\prod_{v=0}^{V-1}E\left\{c_{u,v}{}^2\right\}\right]^{1/U\cdot V}}$$

e.g. U=2 : $$G_{\mathrm{TC,1D}} = \frac{1}{\sqrt{1-\rho^2}}$$

U=V=2, ρ=ρ_h=ρ_v : $$G_{\mathrm{TC,2D}} = \frac{1}{\sqrt[4]{(1+\rho)^2\cdot\left(1-\rho^2\right)^2\cdot(1-\rho)^2}} = \frac{1}{1-\rho^2}$$

Theoretical gain, 1D (11.17) $\Rightarrow$ $G_{\mathrm{1D}} = \dfrac{1}{1-\rho^2}$

Theoretical gain, 2D (11.24) $\Rightarrow$ $G_{\mathrm{2D}} = \dfrac{1}{\left(1-\rho_h{}^2\right)\cdot\left(1-\rho_v{}^2\right)}$

	$G_{\mathrm{TC,1D}}$,U=2	$G_{\mathrm{TC,1D}}$,U=3	$G_{\mathrm{TC,2D}}$,U=V=2	$G_{\mathrm{TC,2D}}$,U=V=3	G_{1D}	G_{2D}
ρ=0.5	1.1547	1.204	1.3333	1.4497	1.3333	1.7777
ρ=0.95	3.2026	4.7125	10.2564	22.208	10.2564	105.194

Other example :
ρ=0.91, G_{1D}=5.8173
U=8 from (4.160) : $G_{TC,1D}$=4.6313
U=2 : $G_{TC,1D}$=2.4119 ; U=3 : $G_{TC,1D}$=3.2253

Problem 12.4

a)

$$\mathbf{I}_1 = \begin{bmatrix} 29 & 4 & 2 & 0 \\ -8 & 0 & 1 & -1 \\ -2 & 2 & -1 & 1 \\ 1 & 5 & 0 & 0 \end{bmatrix} \quad \mathbf{I}_2 = \begin{bmatrix} 59 & 6 & 2 & 0 \\ -11 & 0 & 0 & -1 \\ -2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

$$\mathbf{C}^{(q)}{}_1 = \begin{bmatrix} 232 & 32 & 16 & 0 \\ -64 & 0 & 8 & -8 \\ -16 & 16 & -8 & 8 \\ 8 & 40 & 0 & 0 \end{bmatrix} \quad \mathbf{C}^{(q)}{}_2 = \begin{bmatrix} 236 & 36 & 16 & 0 \\ -66 & 0 & 0 & -16 \\ -16 & 12 & 0 & 0 \\ 0 & 32 & 0 & 0 \end{bmatrix}$$

b) Case 1: 0,0,0,0,1,1,0,0,0,0,0,0,EOB. Case 2: 0,0,0,0,1,2,1,1,EOB

c) Row-wise scan:
Case 1: 11+7+5+1+9+1+3+3+5+5+3+3+3+7+1+1 bit = 68 bit $\Rightarrow$ 4.25 bit/coefficient
Case 2: 13+7+5+1+9+1+1+3+5+3+1+1+1+5+1+1 bit = 58 bit $\Rightarrow$ 3.625 bit/coefficient

d)

$$\mathbf{C} - \mathbf{C}^{(q)}{}_1 = \begin{bmatrix} 235 & 35 & 15 & 3 \\ -67 & 3 & 5 & -9 \\ -17 & 13 & -7 & 9 \\ 5 & 37 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 232 & 32 & 16 & 0 \\ -64 & 0 & 8 & -8 \\ -16 & 16 & -8 & 8 \\ 8 & 40 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 3 & -1 & 3 \\ -3 & 3 & -3 & -1 \\ -1 & -3 & 1 & 1 \\ -3 & -3 & 2 & 1 \end{bmatrix}$$

$$\mathbf{C} - \mathbf{C}^{(q)}{}_2 = \begin{bmatrix} 235 & 35 & 15 & 3 \\ -67 & 3 & 5 & -9 \\ -17 & 13 & -7 & 9 \\ 5 & 37 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 236 & 36 & 16 & 0 \\ -66 & 0 & 0 & -16 \\ -16 & 12 & 0 & 0 \\ 0 & 32 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 & 3 \\ -1 & 3 & 5 & 7 \\ -1 & 1 & -7 & 9 \\ 5 & 5 & 2 & 1 \end{bmatrix}$$

e)

$$\frac{1}{UV} \sum_u \sum_v \left(c_{u,v} - c^{(q)}{}_{1,u,v} \right)^2 = \frac{1}{16} \left[9 \cdot 3^2 + 1 \cdot 2^2 + 6 \cdot 1^2 \right] = 5.6875$$

$$\Rightarrow PSNR = 10 \cdot \log_{10} \frac{255^2}{5.6875} = 40.58 \;\; dB$$

$$\frac{1}{UV} \sum_u \sum_v \left(c_{u,v} - c^{(q)}{}_{2,u,v} \right)^2 = \frac{1}{16} \left[1 \cdot 9^2 + 2 \cdot 7^2 + 3 \cdot 5^2 + 2 \cdot 3^2 + 1 \cdot 2^2 + 7 \cdot 1^2 \right] = 17.6875$$

$$\Rightarrow PSNR = 10 \cdot \log_{10} \frac{255^2}{17.6875} = 35.65 \;\; dB$$

$$R(D) = \frac{1}{2} \log_2 \frac{\sigma_z{}^2}{D} \quad \Rightarrow R(D_1) - R(D_2) = \frac{1}{2} \log_2 \left[\frac{D_2}{D_1} \right]$$

With the difference by distortion, theoretically a rate difference of 0.8184 bit could be expected. The actual difference is only 0.625 bit, which can be explained by the fact that the frequency weighting is suboptimum when the distortion criterion is the unweighted squared error.

Problem 12.5

a) 1D : $\sigma_e^{\ 2} = \sigma_x^{\ 2} \cdot \left(1-\rho^2\right) = 4 \cdot (1-\tfrac{3}{4}) = 1 \quad \Rightarrow G = \dfrac{\sigma_x^{\ 2}}{\sigma_e^{\ 2}} = 4$

2D : $\sigma_e^{\ 2} = \sigma_x^{\ 2} \cdot \left(1-\rho_h^{\ 2}\right) \cdot \left(1-\rho_v^{\ 2}\right) = 4 \cdot (1-\tfrac{3}{4}) \cdot (1-\tfrac{3}{4}) = 0.25 \Rightarrow G = \dfrac{\sigma_x^{\ 2}}{\sigma_e^{\ 2}} = 16$

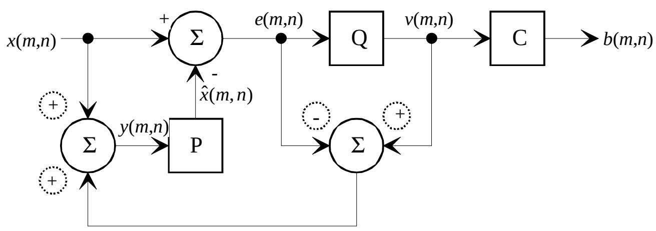
b) 1D : $R_G = -\dfrac{1}{2} \cdot \log_2\left(1-\rho^2\right) = -\dfrac{1}{2} \cdot \log_2(0.25) = 1$ bit

2D : $R_G = -\dfrac{1}{2} \cdot \log_2\left(1-\rho_h^{\ 2}\right) - \dfrac{1}{2} \cdot \log_2\left(1-\rho_v^{\ 2}\right)$

$= 2 \cdot \left(-\dfrac{1}{2} \cdot \log_2(0.25)\right) = 2$ bit

c) At the input of the predictor P, the reconstruction signal $y(m,n)$ must be available. This can be computed from the original signal $x(m,n)$ by adding the quantization error:

$$y(m,n) = x(m,n) - q(m,n) = x(m,n) - \left[e(m,n) - v(m,n)\right] = x(m,n) - e(m,n) + v(m,n)$$



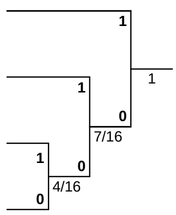
Problem 12.6

a) H=1 in case P(S)=P(W)=0.5; maximum number of bits 16·1=16.

b) $P(\mathrm{S}) = 0.25,\ P(\mathrm{W}) = 0.75 \Rightarrow\ H = -\tfrac{1}{4}\log_2 \tfrac{1}{4} - \tfrac{3}{4}\log_2 \tfrac{3}{4} = -\log_2 \tfrac{1}{4} - \tfrac{3}{4}\log_2 3 \approx 2 - \tfrac{3}{4} \cdot 1.6 = 0.8$

c) P(S,S)=1/16, P(S,W)=P(W,S)=3/16, P(W,W)=9/16

# bits	Symbol	Probability
1	W,W	9/16
2	W,S	3/16
3	S,W	3/16
3	S,S	1/16



$$R=\frac{1}{2}\left(\frac{9}{16}\cdot 1+\frac{3}{16}\cdot 2+\frac{3}{16}\cdot 3+\frac{1}{16}\cdot 3\right)=\frac{1}{2}\cdot\frac{27}{16}=\frac{27}{32}$$

d) i) P(W,W)=5/8, P(S,W)=P(W,S)=P(S,S)=1/8
ii) P(W,W)=1/2, P(S,W)=1/2, P(W,S)=P(S,S)=0

e) i) 5+2+3+3 bit for the code +1 bit signalization = 14 bit
ii) 4+8 bit for the code +1 bit signalization = 13 bit (better)
Advantage of ii) would e.g. not apply when the Huffman code from c) is used, as the combination (S,W) is encoded by 3 bit.

Problem 12.7

a)
$$\sigma_z^2=\sigma_x^2\cdot\left(1-\rho^2\right)=16\cdot(1-\tfrac{9}{16})=7\Rightarrow R(D_{\min})$$
$$=\tfrac{1}{2}\log_2\frac{\sigma_z^2}{D_{\min}}=2\Rightarrow\frac{\sigma_z^2}{D_{\min}}=2^4\Rightarrow D_{\min}=\tfrac{7}{16}$$
$$G_{opt}=\frac{\sigma_x^2}{\sigma_z^2}=\frac{16}{7}$$

b)
$$E\left\{c_0^2\right\}=\left(\frac{\sqrt{2}}{2}\right)^2E\left\{\left[x(n)+x(n-1)\right]^2\right\}=\left(\frac{\sqrt{2}}{2}\right)^2\sigma_x^2\left[2+2\rho\right]=\sigma_x^2\cdot(1+\rho)=16\cdot\tfrac{7}{4}=28$$
$$E\left\{c_1^2\right\}=\left(\frac{\sqrt{2}}{2}\right)^2E\left\{\left[x(n)-x(n-1)\right]^2\right\}=\left(\frac{\sqrt{2}}{2}\right)^2\sigma_x^2\left[2-2\rho\right]=\sigma_x^2\cdot(1-\rho)=16\cdot\tfrac{1}{4}=4$$

Coding gain $G_2=\dfrac{\frac{1}{2}\cdot(28+4)}{\sqrt{28\cdot 4}}=\dfrac{16}{\sqrt{7\cdot 16}}\Rightarrow\dfrac{G_{opt}}{G_2}=\sqrt{\dfrac{16}{7}}\Rightarrow D_{2bit}=D_{\min}\cdot\sqrt{\dfrac{16}{7}}=\sqrt{\dfrac{7}{16}}$

c) i) optimum: $R=\dfrac{1}{2}\log_2\dfrac{\sigma_z^2}{D}=\dfrac{1}{2}\log_2 7=1{,}4$

ii) using **T** :
$$R=\tfrac{1}{2}\cdot\left[\tfrac{1}{2}\log_2\frac{E\{c_0^2\}}{D}+\tfrac{1}{2}\log_2\frac{E\{c_1^2\}}{D}\right]=\tfrac{1}{4}\left[\log_2 28+\log_2 4)\right]=\tfrac{1}{4}\left[\log_2 7+2+2)\right]=1{,}7$$

d)
$$E\left\{c_{00}^2\right\}=\sigma_x^2\cdot(1+\rho)^2=49\quad;\quad E\left\{c_{01}^2\right\}=E\left\{c_{10}^2\right\}=\sigma_x^2\cdot(1+\rho)\cdot(1-\rho)=7$$
$$E\left\{c_{11}^2\right\}=\sigma_x^2\cdot(1-\rho)^2=1\Rightarrow G_{2,2}=\frac{\frac{1}{4}\cdot(49+7+7+1)}{\sqrt[4]{49\cdot 7\cdot 7\cdot 1}}=\frac{16}{7}$$

Problem 13.1

a) (13.11) $\Rightarrow S_{ee}(\Omega_1,\Omega_2)=2\cdot S_{xx}(\Omega_1,\Omega_2)\cdot\left[1-\mathrm{Re}\left\{\mathscr{F}\left(\mathrm{p}(k_e,l_e)\right)\right\}\right]$

$$\mathscr{F}\{\mathrm{p}(k_e)\}=\mathscr{F}\{\mathrm{p}(l_e)\}=\frac{1}{3}\cdot\left[e^{j\Omega_i}+1+e^{-j\Omega_i}\right]=\frac{1}{3}\cdot\left[1+2\cos\Omega_i\right];\ i=1,2$$
$$\Rightarrow S_{ee}(\Omega_1,\Omega_2)=2\cdot S_{xx}(\Omega_1,\Omega_2)\cdot\left[1-\frac{1}{9}(1+2\cos\Omega_1)\cdot(1+2\cos\Omega_2)\right]$$

b) Condition in the stop band of the filter: $S_{xx}(\Omega_1) < S_{ee}(\Omega_1)$.
Cutoff frequency $\Omega_1 : S_{xx}(\Omega_1) = S_{ee}(\Omega_1)$

$$S_{xx}(\Omega_1) = 2 \cdot S_{xx}(\Omega_1) \cdot \left[1 - \frac{1}{3}\left(1 + 2\cos\Omega_1\right)\right] = \frac{4}{3} \cdot S_{xx}(\Omega_1) \cdot \left[1 - \cos\Omega_1\right]$$

$$\Rightarrow 1 = \frac{4}{3} - \frac{4}{3} \cdot \cos\Omega_1 \Rightarrow \frac{1}{3} = \frac{4}{3}\cos\Omega_1 \Rightarrow \cos\Omega_1 = \frac{1}{4} \Rightarrow \Omega_1 \approx 0.42 \cdot \pi$$

c) $$\sigma_x^{\,2} = 2 \cdot \frac{1}{2\pi} \cdot A \cdot \int_0^{\pi} \left(\pi - \Omega_1\right) d\Omega_1 = A \cdot \left[\pi - \frac{1}{\pi} \cdot \frac{\pi^2}{2}\right] = \frac{A\pi}{2} = 1.571 \cdot A$$

$$\sigma_e^{\,2} = 2 \cdot \frac{1}{2\pi} \cdot A \cdot \frac{4}{3} \cdot \int_0^{\pi} \left(\pi - \Omega_1\right) \cdot \left[1 - \cos\Omega_1\right] d\Omega_1 = \frac{4A}{3} \cdot \left[\frac{\pi}{2} - \int_0^{\pi} \cos\Omega_1 + \frac{1}{\pi}\int_0^{\pi} \Omega_1 \cdot \cos\Omega_1\right]$$

$$= \frac{4A}{3} \cdot \left[\frac{\pi}{2} - \left[\sin\Omega_1\right]_0^{\pi} + \frac{1}{\pi}\left[\cos\Omega_1 + \Omega_1 \cdot \sin\Omega_1\right]_0^{\pi}\right] = \frac{4A}{3} \cdot \left[\frac{\pi}{2} - \frac{2}{\pi}\right] = \frac{2A \cdot \left(\pi^2 - 4\right)}{3\pi} = 1.246 \cdot A$$

$$\Rightarrow G = \frac{\sigma_x^{\,2}}{\sigma_e^{\,2}} \approx 1.26\ ;\ G_{SNR} = 10 \cdot \log_{10} G \approx 1.007\ \ dB$$

$$\sigma^2_{e,\,filt} = 2 \cdot \frac{1}{2\pi} \cdot A \cdot \left[\frac{4}{3} \cdot \int_0^{0.42\pi} \left(\pi - \Omega_1\right) \cdot \left[1 - \cos\Omega_1\right] d\Omega_1 + \int_{0.42\pi}^{\pi} \left(\pi - \Omega_1\right) d\Omega_1\right]$$

$$= \frac{A}{\pi} \cdot \left[\frac{4}{3} \cdot \left[\pi \cdot \Omega_1 - \frac{\Omega_1^{\,2}}{2} - \pi \cdot \sin\Omega_1 + \cos\Omega_1 + \Omega_1 \cdot \sin\Omega_1\right]_0^{0.42\pi} + \left[\pi \cdot \Omega_1 - \frac{\Omega_1^{\,2}}{2}\right]_{0.42\pi}^{\pi}\right]$$

$$= \frac{A}{\pi} \cdot \left[\frac{4}{3} \cdot \left[0.42 \cdot \pi^2 - \frac{0.42^2}{2} \cdot \pi^2 - \pi \cdot \sin\left(0.42\pi\right) + \cos\left(0.42\pi\right) + \left(0.42\pi\right) \cdot \sin\left(0.42\pi\right) - 1\right]\right.$$

$$\left. + \left[\frac{\pi^2}{2} - 0.42\pi^2 + \frac{0.42^2}{2}\pi^2\right]\right]$$

$$= 0.850 \cdot A$$

$$\Rightarrow G = \frac{\sigma_x^{\,2}}{\sigma^2_{e,\,filt}} \approx 1.85\ ;\ G_{SNR} = 10 \cdot \log_{10} G \approx 2.668\ \ dB$$

Problem 13.2

a) Vector $\mathbf{k}_1$: $10^2 + 10^2 + 0^2 + (-10)^2 = 300$

Vector $\mathbf{k}_2$: $20^2 + 20^2 + 20^2 + 20^2 = 1600$

Vector $\mathbf{k}_1$ is the better choice according to this criterion.

b) $$\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \cdot \begin{bmatrix} 10 & 10 \\ 0 & -10 \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} 5 \cdot \sqrt{2} & 0 \\ 5 \cdot \sqrt{2} & 10 \cdot \sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 15 & -5 \end{bmatrix}$$

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \cdot \begin{bmatrix} 20 & 20 \\ 20 & 20 \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} 20 \cdot \sqrt{2} & 20 \cdot \sqrt{2} \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} 40 & 0 \\ 0 & 0 \end{bmatrix}$$

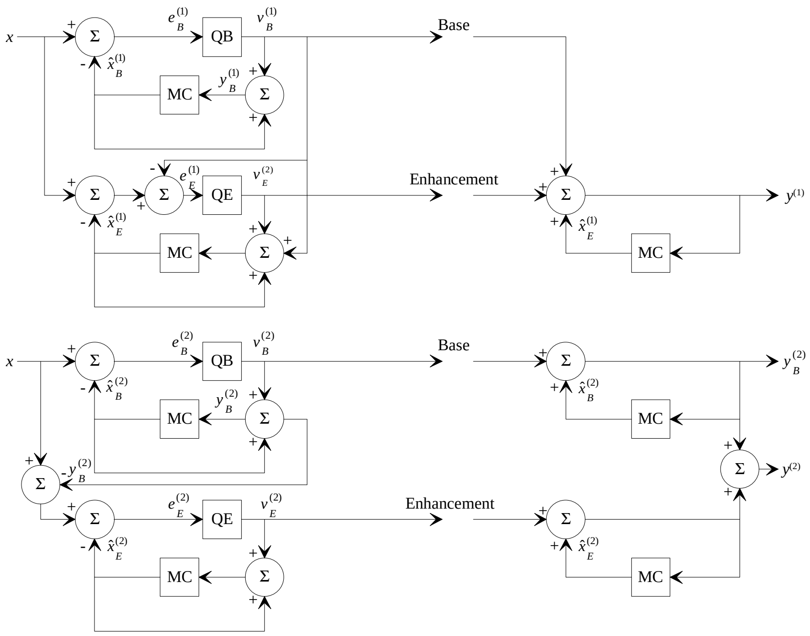
Bit allocation in case of $\mathbf{k}_1$: 7 + 7 + 9 + 7 bit = 30 bit

Bit allocation in case of $\mathbf{k}_2$: 13 + 1 + 1 + 1 bit = 13 bit

In both cases lossless coding of the (discrete) prediction error signal. Choosing $\mathbf{k}_2$ gives a much lower number of bits, which can be explained by the high spatial correlation of the prediction error signal.

Problem 13.3

In the following block diagram the notation definitions used for the signals are given. For the top figure, the signals are marked by superscript "(1)", for the bottom figure, superscript "(2)" is used.



All the subsequent formulations are made in the spectral domain, where the motion-compensated predictor is defined as a linear system of transfer function $H_{\mathrm{MC}}(\mathbf{\Omega})$. The base layer is identical for both structures. Hence, the following relationships hold:

$$\hat{X}_B^{(1)}(\mathbf{\Omega}) = \hat{X}_B^{(2)}(\mathbf{\Omega}) = \hat{X}_B(\mathbf{\Omega}) \quad ; \quad E_B^{(1)}(\mathbf{\Omega}) = E_B^{(2)}(\mathbf{\Omega}) = E_B(\mathbf{\Omega})$$

$$V_B^{(1)}(\mathbf{\Omega}) = V_B^{(2)}(\mathbf{\Omega}) = V_B(\mathbf{\Omega}) \quad ; \quad Y_B^{(1)}(\mathbf{\Omega}) = Y_B^{(2)}(\mathbf{\Omega}) = Y_B(\mathbf{\Omega})$$

Further, as both reconstructed signals shall be equal:

$$Y(\mathbf{\Omega}) = Y^{(1)}(\mathbf{\Omega}) = \left[V_B(\mathbf{\Omega}) + V_E^{(1)}(\mathbf{\Omega})\right] \cdot \frac{1}{1 - H_{MC}(\mathbf{\Omega})}$$

$$= Y^{(2)}(\mathbf{\Omega}) = V_B(\mathbf{\Omega}) \cdot \frac{1}{1 - H_{MC}(\mathbf{\Omega})} + V_E^{(2)}(\mathbf{\Omega}) \cdot \frac{1}{1 - H_{MC}(\mathbf{\Omega})}$$

$$\Rightarrow V_E^{(1)}(\mathbf{\Omega}) = V_E^{(2)}(\mathbf{\Omega}) = V_E(\mathbf{\Omega})$$

This leads to the following definitions of prediction estimates:

$$\hat{X}_E^{(1)}(\mathbf{\Omega}) = H_{MC}(\mathbf{\Omega}) \cdot \frac{V_B(\mathbf{\Omega}) + V_E(\mathbf{\Omega})}{1 - H_{MC}(\mathbf{\Omega})}$$

$$\hat{X}_E^{(2)}(\mathbf{\Omega}) = H_{MC}(\mathbf{\Omega}) \cdot \frac{V_E(\mathbf{\Omega})}{1 - H_{MC}(\mathbf{\Omega})}$$

$$\Rightarrow \hat{X}_E^{(1)}(\mathbf{\Omega}) - \hat{X}_E^{(2)}(\mathbf{\Omega}) = H_{MC}(\mathbf{\Omega}) \cdot \frac{V_B(\mathbf{\Omega})}{1 - H_{MC}(\mathbf{\Omega})} = H_{MC}(\mathbf{\Omega}) \cdot Y_B(\mathbf{\Omega})$$

Finally, the prediction errors at the enhancement quantizer inputs are:

$$\begin{aligned}
E_E^{(1)}(\mathbf{\Omega}) &= X(\mathbf{\Omega}) - \hat{X}_E^{(1)}(\mathbf{\Omega}) - V_B(\mathbf{\Omega}) \\
&= X(\mathbf{\Omega}) - H_{MC}(\mathbf{\Omega}) \cdot \frac{V_B(\mathbf{\Omega}) + V_E(\mathbf{\Omega})}{1 - H_{MC}(\mathbf{\Omega})} - V_B(\mathbf{\Omega}) \\
&= X(\mathbf{\Omega}) - H_{MC}(\mathbf{\Omega}) \cdot \frac{V_E(\mathbf{\Omega})}{1 - H_{MC}(\mathbf{\Omega})} - V_B(\mathbf{\Omega}) \cdot \underbrace{\left[1 + \frac{H_{MC}(\mathbf{\Omega})}{1 - H_{MC}(\mathbf{\Omega})}\right]}_{=\frac{1}{1 - H_{MC}(\mathbf{\Omega})}} \\
&= X(\mathbf{\Omega}) - H_{MC}(\mathbf{\Omega}) \cdot \frac{V_E(\mathbf{\Omega})}{1 - H_{MC}(\mathbf{\Omega})} - Y_B(\mathbf{\Omega})
\end{aligned}$$

$$\begin{aligned}
E_E^{(2)}(\mathbf{\Omega}) &= X(\mathbf{\Omega}) - Y_B(\mathbf{\Omega}) - \hat{X}_E^{(2)}(\mathbf{\Omega}) - Y_B(\mathbf{\Omega}) \\
&= X(\mathbf{\Omega}) - H_{MC}(\mathbf{\Omega}) \cdot \frac{V_E(\mathbf{\Omega})}{1 - H_{MC}(\mathbf{\Omega})} - Y_B(\mathbf{\Omega}) \\
&\Rightarrow E_E^{(1)}(\mathbf{\Omega}) = E_E^{(2)}(\mathbf{\Omega}) = E_E(\mathbf{\Omega})
\end{aligned}$$

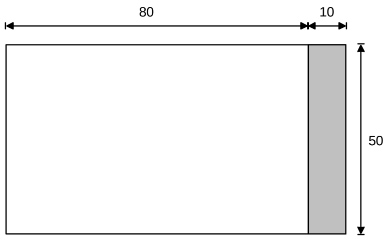
Problem 13.4

a) $\frac{1}{2}\log_2 \frac{\sigma_x^2}{D} = 6 \Rightarrow D = \frac{\sigma_x^2}{2^{12}} = \frac{1}{256}$

b) $R_{\mathrm{G,1D}} = -\frac{1}{2}\log_2(1-\rho^2) = \frac{1}{2}\log_2(4) = 1 bit$

$R_{\mathrm{G,2D}} = 2R_{\mathrm{G,1D}} = 2 bit$

c) The shaded area is uncovered and must be newly encoded. This is 1/9 of the total area, and hence requires 1/9 of the bit number that would be necessary to encode a complete frame by intraframe coding.



d) Number of pixels to be encoded: 90x50 (1st frame) + 90x50x1/9 x9 (subsequent frames) =9000; 9000 pixel x 4 bit/pixel = 36000 bit.

e) 4500 pixel in 1st frame $\Rightarrow$ 9000/4500=2 bit/pixel. Considering the coding gain by utilizing the spatial correlation:

$$D_1 = \frac{\sigma_x^{\,2}}{2^{2R_1}} \cdot (1-\rho_h^{\,2})(1-\rho_v^{\,2}) = \frac{\sigma_x^{\,2}}{2^4} \cdot \frac{1}{16} = \frac{1}{16} \Rightarrow \frac{D_1}{D} = \frac{\frac{1}{16}}{\frac{1}{256}} = 16$$

f) Coding distortion from the previous frame is fed into the prediction error signal. The spectrum of the prediction error signal then is, according to (13.12) and (13.13) $S_{ee}(\Omega_1,\Omega_2)=\min(S_{xx}(\Omega_1,\Omega_2),\Theta)$

The coding distortion from e) is white noise, as according to the following estimation condition (11.23) still holds, i.e. encoding is performed within the range of "high distortion":

$$D_1 \overset{?}{\leq} \frac{(1-\rho_\mathrm{h})\cdot(1-\rho_\mathrm{v})}{(1+\rho_\mathrm{h})\cdot(1+\rho_\mathrm{v})} \cdot 16 = \frac{\left(1-\sqrt{3}/2\right)^2}{\left(1+\sqrt{3}/2\right)^2} \cdot 16 = \frac{7/4-\sqrt{3}}{7/4+\sqrt{3}} \cdot 16$$

$$= \left(7/4-\sqrt{3}\right)^2 \cdot 256 \approx 0.0004 \cdot 256 \approx 0.1 > \frac{1}{16}$$

Hence in this case, $\Theta=D_1$ and $S_{ee}(\Omega_1,\Omega_2)=D_1$=const.