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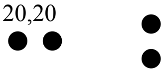
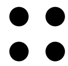
Multimedia Content Analysis

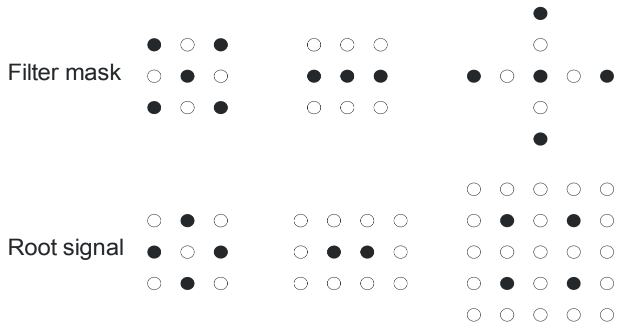
Solutions to End-of-chapter Problems

Problem 2.1

a) i) 20,20 ii) 15,20 iii) 25,25 iv) 20,20

b) iii) Two different root signals:  iv): 

Problem 2.2



Problem 2.3

With assumption of constant value extension outside of the marked bounding rectangles :

a) $\mathbf{G} = \begin{bmatrix} 10 & 10 & 10 & 10 & 20 & 20 & 20 \\ 10 & 10 & 10 & 10 & 20 & 20 & 20 \\ 10 & 10 & 10 & 10 & 20 & 20 & 20 \\ 10 & 10 & 10 & 10 & 20 & 20 & 20 \\ 10 & 10 & 10 & 10 & 20 & 20 & 20 \end{bmatrix}$

b) $\mathbf{G} = \begin{bmatrix} 10 & 10 & 20 & 20 & 20 & 20 & 20 \\ 10 & 10 & 20 & 20 & 20 & 20 & 20 \\ 10 & 10 & 20 & 20 & 20 & 20 & 20 \\ 10 & 10 & 10 & 20 & 20 & 20 & 20 \\ 10 & 10 & 10 & 20 & 20 & 20 & 20 \end{bmatrix}$

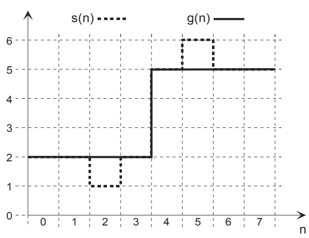
c) $\mathbf{G} = \begin{bmatrix} 10 & 10 & 10 & 10 & 10 & 20 & 20 \\ 10 & 10 & 10 & 10 & 10 & 20 & 20 \\ 10 & 10 & 10 & 10 & 10 & 10 & 20 \\ 10 & 10 & 10 & 10 & 10 & 10 & 20 \\ 10 & 10 & 10 & 10 & 10 & 10 & 20 \end{bmatrix}$

d) $\mathbf{G} = \begin{bmatrix} 0 & 0 & 10 & 10 & 10 & 0 & 0 \\ 0 & 0 & 10 & 10 & 10 & 0 & 0 \\ 0 & 0 & 10 & 10 & 10 & 10 & 0 \\ 0 & 0 & 0 & 10 & 10 & 10 & 0 \\ 0 & 0 & 0 & 10 & 10 & 10 & 0 \end{bmatrix}$

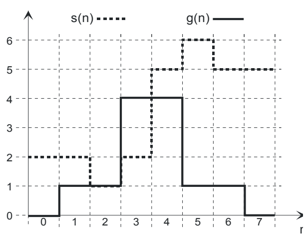
e) $\mathbf{G} = \begin{bmatrix} 10 & 10 & 10 & 10 & 20 & 20 & 20 \\ 10 & 10 & 10 & 10 & 20 & 20 & 20 \\ 10 & 10 & 10 & 10 & 20 & 20 & 20 \\ 10 & 10 & 10 & 10 & 10 & 20 & 20 \\ 10 & 10 & 10 & 10 & 10 & 20 & 20 \end{bmatrix}$

f) $\mathbf{G} = \begin{bmatrix} 10 & 10 & 10 & 20 & 20 & 20 & 20 \\ 10 & 10 & 10 & 20 & 20 & 20 & 20 \\ 10 & 10 & 10 & 10 & 20 & 20 & 20 \\ 10 & 10 & 10 & 10 & 20 & 20 & 20 \\ 10 & 10 & 10 & 10 & 20 & 20 & 20 \end{bmatrix}$

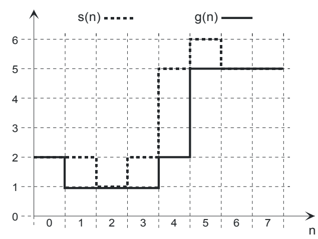
Problem 2.4



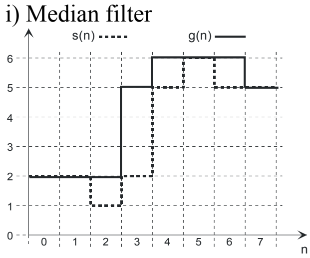
i) Median filter



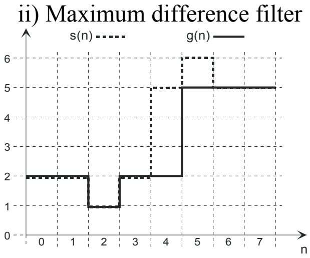
ii) Maximum difference filter



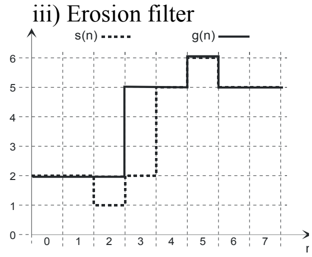
iii) Erosion filter



iv) Dilation filter



v) Opening filter



vi) Closing filter

Problem 2.5

a) A=med[2,3,5]=3, B=med[3,4,5]=4, C=med[2,3,4]=3, D=med[2,4,5]=4

b) Distance parameters for bilinear interpolation: A – d_1=.25, d_2=.25 ; B – d_1=.25, d_2=.75.

$$\hat{A}=\frac{9}{16}\cdot 3+\frac{3}{16}\cdot 2+\frac{3}{16}\cdot 5+\frac{1}{16}\cdot 4=\frac{52}{16}=3.25$$

$$\hat{B}=\frac{3}{16}\cdot 3+\frac{1}{16}\cdot 2+\frac{9}{16}\cdot 5+\frac{3}{16}\cdot 4=\frac{68}{16}=4.25$$

Deviation between median and linear interpolation is .25 in both cases.

Problem 2.6

With the basis function centered around $t(n)$, $t'=[t-t(n)]/T$, validity for interpolation within range $|t'|\leq 1/2$.

a)

$$\hat{s}(t')=c(n-1)\frac{\left(1.5-(t'+1)\right)^2}{2}+c(n)\left(\frac{3}{4}-t'^2\right)+c(n+1)\frac{\left(1.5-(1-t')\right)^2}{2}$$

$$=\frac{1}{8}\left[c(n-1)\left(8t'^2-8t'+1\right)+c(n)\left(6-8t'^2\right)+c(n+1)\left(8t'^2+8t'+1\right)\right]=\frac{1}{8}\begin{bmatrix}t'^2 & t' & 1\end{bmatrix}\begin{bmatrix}8 & -8 & 8\\ -8 & 0 & 8\\ 1 & 6 & 1\end{bmatrix}\begin{bmatrix}c(n-1)\\ c(n)\\ c(n+1)\end{bmatrix}$$

$$\hat{s}\left[t(n)\right]=\frac{1}{8}\left[c(n-1)+6c(n)+c(n+1)\right]$$

b) with t'=0 :

$$\Rightarrow\begin{bmatrix}c(0)\\ c(1)\\ c(2)\\ \vdots\\ \\ c(M-1)\end{bmatrix}=8\begin{bmatrix}6 & 1 & 0 & & \cdots & 1\\ 1 & 6 & 1 & 0 & & \vdots\\ 0 & 1 & 6 & 1 & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & \\ \vdots & & 0 & 1 & 6 & 1\\ 1 & \cdots & & 0 & 1 & 6\end{bmatrix}^{-1}\cdot\begin{bmatrix}s(0)\\ s(1)\\ s(2)\\ \vdots\\ \\ s(M-1)\end{bmatrix}$$

Problem 3.1

The 'generating function' (3.9) defines cumulants as weights of an infinite series, where for the case of stationarity:

$$v_s(\omega)=\log\mathcal{E}\left\{e^{\omega s(t)}\right\}=\sum_{P=1}^{\infty}\kappa_s^{(P)}\frac{\omega^P}{P!}\quad\text{with}\quad\kappa_s^{(P)}=\left.\frac{\partial^P}{\partial\omega^P}v_s(\omega)\right|_{\omega=0}$$

For case of stationary process with Gaussian PDF, $v_s(\omega)=\log\int_{-\infty}^{\infty}\mathrm{e}^{\omega x}p_s(x)\mathrm{d}x$, where the integral can be interpreted as a Fourier integral, for which the solution is again a Gaussian function. Therefore, $v_s(\omega)=m_s\omega+\sigma_s^2\frac{\omega^2}{2}$. Consequently, $\kappa_s^{(1)}=m_s$, $\kappa_s^{(2)}=\sigma_s^2$ and $\kappa_s^{(P)}=0$ for $P>2$. With the relation between cumualants and moments (3.12), we therefore get in this case for P>2 $m_s^{(P)}=\sum_{p=1}^{2}\binom{P-1}{p-1}\kappa_s^{(p)}m_s^{(P-p)}$, such that all higher order moments can be recursively computed from moments of orders 1 and 2 (or mean and variance).

Problem 3.2

a)

$$H(z_1)=\frac{1}{3}\left[z_1^1+1+z_1^{-1}\right]$$

$$H(f_1)=\frac{1}{3}\left[\mathrm{e}^{\mathrm{j}2\pi f_1}+1+\mathrm{e}^{-\mathrm{j}2\pi f_1}\right]=\frac{1}{3}\left[1+2\cos\left(2\pi f_1\right)\right]$$

b) $1+2\cos(2\pi f_1)=0$ for $\cos(2\pi f_1)=-1/2$ ⇒ Zeros of Fourier transfer function at $f_1=1/2\pm 1/6$

Inverse filter : $H^{\mathrm{I}}(f_1)=\dfrac{3}{1+2\cos\left(2\pi f_1\right)}$

Pseudo inverse filter : $H^{\mathrm{PI}}(f_1)=\begin{cases}\dfrac{3}{1+2\cos\left(2\pi f_1\right)} & \text{for}\quad f_1\neq\dfrac{1}{2}\pm\dfrac{1}{1}\\ 0\ \text{ for }\ f_1=\dfrac{1}{2}\pm\dfrac{1}{1}.\end{cases}$

Discrete frequency lines of DFT at $f_1(k_1)=\dfrac{k_1}{M_1}$

$H^{\mathrm{I}}(k_1) = \dfrac{3}{1+\cos\dfrac{2\pi k_1}{M_1}}$; critical cases are $\dfrac{k_1}{M_1} = \dfrac{1}{2} \pm \dfrac{1}{6}$ for integer k_1.

Inverse filtering unstable for k_1=10, k_1=20 with M_1=30; no instability for M_1=32.

c) Power spectrum of the attenuated signal $\Phi_{gg,\delta}(f_1) = \dfrac{A^2}{9}\left[1+2\cos(2\pi f_1)\right]^2$

Power spectrum of disturbing noise $\Phi_{vv,\delta}(f_1) = \dfrac{A^2}{4}$

Condition $\dfrac{A^2}{9}\left[1+2\cos(2\pi f_1)\right]^2 \geq \dfrac{A^2}{4}$

$1+4\cos(2\pi f_1)+4\cos^2(2\pi f_1) \geq \dfrac{9}{4} \Rightarrow \cos^2(2\pi f_1)+\cos(2\pi f_1)-\dfrac{5}{16} \geq 0$

Limit case for $\cos(2\pi f_1) = -\dfrac{1}{2} \pm \sqrt{\dfrac{1}{4}+\dfrac{5}{16}} = -\dfrac{1}{2} \pm \dfrac{3}{4}$, only the non-transcendent case with |cos|<1 is regarded here ⇒ arc-cos(1/4)≈0.4196π:

$$\Rightarrow H^{\mathrm{PI}}(f_1) = \begin{cases} \dfrac{3}{1+2\cos(2\pi f_1)} & \text{for } f_1 \leq 0.2098 \\ 0 \text{ for } f_1 > 0.2098 \end{cases}$$

d) Wiener filter:

$$H^{\mathrm{I}}(f_1) = \frac{H^*(f_1)}{|H(f_1)|^2 + \dfrac{\Phi_{vv,\delta}(f_1)}{\Phi_{ss,\delta}(f_1)}} = \frac{\frac{1}{3}\left[1+2\cos(2\pi f_1)\right]}{\frac{1}{9}\left[1+2\cos(2\pi f_1)\right]^2 + \dfrac{A^2/4}{A^2}} = \frac{1+2\cos(2\pi f_1)}{\frac{1}{3}\left[1+2\cos(2\pi f_1)\right]^2 + \frac{3}{4}}$$

Attenuation for f_1=0.2098 : $\dfrac{1+2\cdot\frac{1}{4}}{\frac{1}{3}\left[1+2\cdot\frac{1}{4}\right]^2+\frac{3}{4}} = 1$

Problem 3.3

a) $H(f) = \frac{1}{4}\cdot\left(\mathrm{e}^{\mathrm{j}2\pi f}+2+\mathrm{e}^{-\mathrm{j}2\pi f}\right) = \frac{1}{2}\cdot\left(1+\cos(2\pi f)\right)$

b) $H^{\mathrm{I}}(f) = \dfrac{1}{H(f)} = \dfrac{2}{1+\cos(2\pi f)}$. Zero for cos(2$\pi f$)=–1, i.e. f=1/2 (half sampling frequency). No pseudo inverse filter necessary, if sampling theorem is observed.

c) $H^{\mathrm{IW}}(j\Omega) = \dfrac{\frac{1}{2}\left(1+\cos(2\pi f)\right)}{\frac{1}{4}\left(1+\cos(2\pi f)\right)^2+\frac{1}{4}\cdot\sin^2(2\pi f)} = \dfrac{\frac{1}{2}\left(1+\cos(2\pi f)\right)}{\frac{1}{4}+\frac{1}{2}\cos(2\pi f)+\frac{1}{4}\cos^2(2\pi f)+\frac{1}{4}\cdot\sin^2(2\pi f)} = 1$

d) Substitution $|f| = \sqrt{f_1^2+f_2^2}$: $H_{2\mathrm{D}}(f_1,f_2) = \frac{1}{2}\cdot\left(1+\cos 2\pi\sqrt{f_1^2+f_2^2}\right)$

e) Transfer function =0 for $\sqrt{f_1^2+f_2^2} = 1/2$, pseudo inverse filter is necessary here:

$$H_{2\mathrm{D}}^{\mathrm{IP}}(f_1,f_2) = \begin{cases} \dfrac{2}{1+\cos 2\pi\sqrt{f_1^2+f_2^2}}, & \text{if } \sqrt{f_1^2+f_2^2} \neq 1/2 \\ 0, & \text{if } \sqrt{f_1^2+f_2^2} = 1/2 \end{cases}$$

Note: Better define $H^{\mathrm{IP}}(f_1,f_2)=0$ for $\sqrt{f_1^2+f_2^2} > 1/2$, since in this range an alias component of the original 1D filter would be defined as frequency transfer function.

Problem 3.4

a) Case i) $\mathbf{g} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 9 \\ 15 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \end{bmatrix}$ Case ii) $\mathbf{g} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 9 \\ 15 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 11 \\ 15 \end{bmatrix}$

b) Case i) according to (3.43), L>K : $\mathbf{H}^{\mathrm{p}} = \mathbf{H}^{\mathrm{T}} \cdot \left(\mathbf{H}\mathbf{H}^{\mathrm{T}}\right)^{-1}$

$$\left(\mathbf{HH}^{\mathrm{T}}\right)^{-1}=\left(\begin{bmatrix}\frac{1}{2}&\frac{1}{2}&0\\0&\frac{1}{2}&\frac{1}{2}\end{bmatrix}\begin{bmatrix}\frac{1}{2}&0\\\frac{1}{2}&\frac{1}{2}\\0&\frac{1}{2}\end{bmatrix}\right)^{-1}=\begin{bmatrix}\frac{1}{2}&\frac{1}{4}\\\frac{1}{4}&\frac{1}{2}\end{bmatrix}^{-1}=\begin{bmatrix}\frac{8}{3}&-\frac{4}{3}\\-\frac{4}{3}&\frac{8}{3}\end{bmatrix}$$

$$\Rightarrow \mathbf{H}^{\mathrm{p}}=\begin{bmatrix}\frac{1}{2}&0\\\frac{1}{2}&\frac{1}{2}\\0&\frac{1}{2}\end{bmatrix}\cdot\begin{bmatrix}\frac{8}{3}&-\frac{4}{3}\\-\frac{4}{3}&\frac{8}{3}\end{bmatrix}=\begin{bmatrix}\frac{4}{3}&-\frac{2}{3}\\\frac{2}{3}&\frac{2}{3}\\-\frac{2}{3}&\frac{4}{3}\end{bmatrix}$$

Case ii) according to (3.43), $L<K$: $\mathbf{H}^{\mathrm{p}}=\left(\mathbf{H}^{\mathrm{T}}\mathbf{H}\right)^{-1}\cdot\mathbf{H}^{\mathrm{T}}$

$$\left(\mathbf{H}^{\mathrm{T}}\mathbf{H}\right)^{-1}=\left(\begin{bmatrix}1&\frac{1}{3}&0&0\\0&\frac{2}{3}&\frac{2}{3}&0\\0&0&\frac{1}{3}&1\end{bmatrix}\begin{bmatrix}1&0&0\\\frac{1}{3}&\frac{2}{3}&0\\0&\frac{2}{3}&\frac{1}{3}\\0&0&1\end{bmatrix}\right)^{-1}=\begin{bmatrix}\frac{10}{9}&\frac{2}{9}&0\\\frac{2}{9}&\frac{8}{9}&\frac{2}{9}\\0&\frac{2}{9}&\frac{10}{9}\end{bmatrix}^{-1}=\begin{bmatrix}0.95&-0.25&0.05\\-0.25&1.25&-0.25\\0.05&-0.25&0.95\end{bmatrix}$$

$$\Rightarrow \mathbf{H}^{\mathrm{p}}=\begin{bmatrix}0.95&-0.25&0.05\\-0.25&1.25&-0.25\\0.05&-0.25&0.95\end{bmatrix}\cdot\begin{bmatrix}1&\frac{1}{3}&0&0\\0&\frac{2}{3}&\frac{2}{3}&0\\0&0&\frac{1}{3}&1\end{bmatrix}=\begin{bmatrix}0.95&0.15&-0.15&0.05\\-0.25&0.75&0.75&-0.25\\0.05&-0.15&0.15&0.95\end{bmatrix}$$

c) Case i) $\hat{\mathbf{s}}=\begin{bmatrix}\frac{4}{3}&-\frac{2}{3}\\\frac{2}{3}&\frac{2}{3}\\-\frac{2}{3}&\frac{4}{3}\end{bmatrix}\cdot\begin{bmatrix}6\\12\end{bmatrix}=\begin{bmatrix}0\\12\\12\end{bmatrix}$

Case ii) $\hat{\mathbf{s}}=\begin{bmatrix}0.95&0.15&-0.15&0.05\\-0.25&0.75&0.75&-0.25\\0.05&-0.15&0.15&0.95\end{bmatrix}\cdot\begin{bmatrix}3\\7\\11\\15\end{bmatrix}=\begin{bmatrix}3\\9\\15\end{bmatrix}$

d) Eigenvalues of $\mathbf{HH}^{\mathrm{T}}$ (cf. Problem 2.8 of MSC) : $\lambda_0=1/2+1/4=3/4$; $\lambda_1=1/2-1/4=1/4$
$\Rightarrow$ Matrix of singular values (rank of $\mathbf{H}$ is R=2)

$$\mathbf{\Phi}^{\mathrm{T}}\mathbf{H}\mathbf{\Psi}=\mathbf{\Lambda}^{(1/2)}=\begin{bmatrix}\frac{\sqrt{3}}{2}&0&0\\0&\frac{1}{2}&0\end{bmatrix}$$

$$\mathbf{\Phi}^{\mathrm{T}}\left[\mathbf{HH}^{\mathrm{T}}\right]\mathbf{\Phi}=\mathbf{\Phi}^{\mathrm{T}}\begin{bmatrix}\frac{1}{2}&\frac{1}{4}\\\frac{1}{4}&\frac{1}{2}\end{bmatrix}\mathbf{\Phi}=\mathbf{\Lambda}^{(K)}=\begin{bmatrix}\frac{3}{4}&0\\0&\frac{1}{4}\end{bmatrix};\ \mathbf{\Phi}=\begin{bmatrix}\frac{\sqrt{2}}{2}&\frac{\sqrt{2}}{2}\\\frac{\sqrt{2}}{2}&-\frac{\sqrt{2}}{2}\end{bmatrix}$$

$$\mathbf{\Psi}^{\mathrm{T}}\left[\mathbf{H}^{\mathrm{T}}\mathbf{H}\right]\mathbf{\Psi}=\mathbf{\Psi}^{\mathrm{T}}\begin{bmatrix}\frac{1}{4}&\frac{1}{4}&0\\\frac{1}{4}&\frac{1}{2}&\frac{1}{4}\\0&\frac{1}{4}&\frac{1}{4}\end{bmatrix}\mathbf{\Psi}=\mathbf{\Lambda}^{(L)}=\begin{bmatrix}\frac{3}{4}&0&0\\0&\frac{1}{4}&0\\0&0&0\end{bmatrix};\ \mathbf{\Psi}=\begin{bmatrix}\frac{\sqrt{6}}{6}&\frac{\sqrt{2}}{2}&-\frac{\sqrt{3}}{3}\\\frac{\sqrt{6}}{3}&0&\frac{\sqrt{3}}{3}\\\frac{\sqrt{6}}{6}&-\frac{\sqrt{2}}{2}&-\frac{\sqrt{3}}{3}\end{bmatrix}$$

Computation of estimate by generalized inverse from the first singular value:

$$\mathbf{H}_0{}^{\mathrm{g}}=\left(\lambda_0\right)^{-1/2}\boldsymbol{\phi}_0\boldsymbol{\psi}_0{}^{\mathrm{T}}=\frac{2}{\sqrt{3}}\cdot\begin{bmatrix}\frac{\sqrt{6}}{6}\\\frac{\sqrt{6}}{3}\\\frac{\sqrt{6}}{6}\end{bmatrix}\cdot\begin{bmatrix}\frac{\sqrt{2}}{2}&\frac{\sqrt{2}}{2}\end{bmatrix}=\begin{bmatrix}\frac{1}{3}&\frac{1}{3}\\\frac{2}{3}&\frac{2}{3}\\\frac{1}{3}&\frac{1}{3}\end{bmatrix}\Rightarrow\hat{\mathbf{s}}=\begin{bmatrix}\frac{1}{3}&\frac{1}{3}\\\frac{2}{3}&\frac{2}{3}\\\frac{1}{3}&\frac{1}{3}\end{bmatrix}\cdot\begin{bmatrix}6\\12\end{bmatrix}=\begin{bmatrix}6\\12\\6\end{bmatrix}$$

Problem 3.5
Mean and variance of the signal:

$$m_s=0.2\cdot0+0.8\cdot1=0.8$$

$$\sigma_s{}^2=\left[0.2\cdot0^2+0.8\cdot1^2\right]-m_s{}^2=0.8-0.64=0.16$$

The inverse "autocovariance matrices" of size 1x1 relate to variances of the signal and the noise,

$$\mathbf{C}_{ss}^{-1}=\frac{1}{\sigma_s{}^2}\ ;\quad \mathbf{C}_{vv}^{-1}=\frac{1}{\sigma_v{}^2}$$

a) Maximum likelihood criterion (3.65) : Minimization of $\Delta_{\text{ML}} = \dfrac{\left[g-\hat{s}\right]^2}{2\sigma_v^{\,2}}$ gives

	g=0.3	g=0.5	g=0.7
$\hat{s}=0$	0.45	1.25	2.45
$\hat{s}=1$	2.45	1.25	0.45

The decision for the value g=0.5 is not unique; the behavior is equivalent to a threshold decision with Θ=0.5

b) Maximum-a-posteriori criterion according to (3.71) : Minimization of $\Delta_{\text{MAP}} = \dfrac{\left[g-\hat{s}\right]^2}{2\sigma_v^{\,2}} + \dfrac{\left[\hat{s}-m_s\right]^2}{2\sigma_s^{\,2}}$

	g=0.3	g=0.5	g=0.7
$\hat{s}=0$	0.45+2=2.45	1.25+2=3.25	2.45+2
$\hat{s}=1$	2.45+0.125=2.575	1.25+0.125=1.375	0.45+0.125=0.575

The value g=0.5 is now clearly mapped into $\hat{s}=1$; the behavior is equivalent to a threshold decision with

$$\frac{\Theta^2}{2\sigma_v^{\,2}} + 2 = \frac{\left(\Theta-1\right)^2}{2\sigma_v^{\,2}} + 0.125 \Rightarrow \Theta = 0.3125$$

Problem 3.6

a)
$$\mathbf{C}_{vv} = \mathbf{C}_{vv}^{\;-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad ; \quad \mathbf{C}_{ss} = \frac{16}{7}\begin{bmatrix} 1 & 3/4 \\ 3/4 & 1 \end{bmatrix}$$
$$\Rightarrow \mathbf{C}_{ss}^{\;-1} = \frac{7}{16}\cdot\frac{1}{(1-9/16)}\begin{bmatrix} 1 & -3/4 \\ -3/4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3/4 \\ -3/4 & 1 \end{bmatrix}$$

b)
$$\mathbf{g}-\hat{\mathbf{s}}_1 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\text{T}} \quad ; \quad \mathbf{g}-\hat{\mathbf{s}}_2 = \begin{bmatrix} -2 & 0 \end{bmatrix}^{\text{T}}$$
$$\Delta_{\text{ML}}(\hat{\mathbf{s}}_1) = \frac{1}{2}\cdot\begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \end{bmatrix}^{\text{T}} = 1 \quad ; \quad \Delta_{\text{ML}}(\hat{\mathbf{s}}_2) = \frac{1}{2}\cdot\begin{bmatrix} -2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -2 & 0 \end{bmatrix}^{\text{T}} = 2$$

This leads to the conclusion that the first hypothesis is better.

c)
$$\Delta_{\text{MAP}}(\hat{\mathbf{s}}_1) = \frac{1}{2}\cdot\begin{bmatrix} 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & -3/4 \\ -3/4 & 1 \end{bmatrix}\begin{bmatrix} 0 & 4 \end{bmatrix}^{\text{T}} + \frac{1}{2}\cdot\begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \end{bmatrix}^{\text{T}} = 8+1 = 9$$
$$\Delta_{\text{MAP}}(\hat{\mathbf{s}}_2) = \frac{1}{2}\cdot\begin{bmatrix} 3 & 3 \end{bmatrix}\begin{bmatrix} 1 & -3/4 \\ -3/4 & 1 \end{bmatrix}\begin{bmatrix} 3 & 3 \end{bmatrix}^{\text{T}} + \frac{1}{2}\cdot\begin{bmatrix} -2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -2 & 0 \end{bmatrix}^{\text{T}} = 2.25+2 = 4.25$$

Now, the second hypothesis is better, explains by the correlated signal model.

d) The signal variance reciprocally influences the first term of the MAP estimate. Both hypotheses would be equivalent, if

$$\frac{8}{c} + 1 = \frac{2.25}{c} + 2 \quad \Rightarrow c = 5.75$$

For $\sigma_s^2 > c\cdot 16/7$, the first hypothesis is preferred by the MAP estimation: If higher variance of the signal is expected, the high deviation between both values in the vector appears more reasonable.

Problem 4.1

Angle between two K-dimensional vectors **a** and **b** :

$$\cos\phi = \frac{\mathbf{a}^{\text{T}}\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \frac{a_1 b_1 + a_2 b_2 + \ldots + a_K b_K}{\sqrt{a_1^{\,2} + a_2^{\,2} + \ldots + a_K^{\,2}}\cdot\sqrt{b_1^{\,2} + b_2^{\,2} + \ldots + y_K^{\,2}}}$$

a)
$$\varphi(\boldsymbol{\phi}_{\text{Y}}, \boldsymbol{\phi}_{\text{C}_\text{b}}) = \arccos\frac{-0.299\cdot 0.169 - 0.587\cdot 0.331 + 0.114\cdot 0,5}{\sqrt{0.299^2 + 0.587^2 + 0.114^2}\cdot\sqrt{0.169^2 + 0.331^2 + 0.5^2}} = 116.8^{\text{o}}$$

$$\varphi(\phi_Y,\phi_{C_r}) = \arccos\frac{0.299\cdot 0.5 - 0.587\cdot 0.419 - 0.114\cdot 0.081}{\sqrt{0.299^2+0.587^2+0.114^2}\cdot\sqrt{0.5^2+0.419^2+0.081^2}} = 103.9^\circ$$

$$\varphi(\phi_{C_b},\phi_{C_r}) = \arccos\frac{-0.169\cdot 0.5 + 0.331\cdot 0.419 - 0.5\cdot 0.081}{\sqrt{0.169^2+0.331^2+0.5^2}\cdot\sqrt{0.5^2+0.419^2+0.081^2}} = 88.08^\circ$$

b) $$\varphi(\phi_X,\phi_Y) = \arccos\frac{0.607\cdot 0.299 + 0.174\cdot 0.587 + 0.200\cdot 0.114}{\sqrt{0.607^2+0.174^2+0.200^2}\cdot\sqrt{0.299^2+0.587^2+0.114^2}} = 46.2^\circ$$

$$\varphi(\phi_X,\phi_Z) = \arccos\frac{0.607\cdot 0.000 + 0.174\cdot 0.066 + 0.200\cdot 1.116}{\sqrt{0.607^2+0.174^2+0.200^2}\cdot\sqrt{0.000^2+0.066^2+1.116^2}} = 71.52^\circ$$

$$\varphi(\phi_Y,\phi_Z) = \arccos\frac{0.299\cdot 0.000 + 0.587\cdot 0.066 + 0.114\cdot 1.116}{\sqrt{0.299^2+0.587^2+0.114^2}\cdot\sqrt{0.000^2+0.066^2+1.116^2}} = 77.17^\circ$$

c) $$\varphi(\phi_I,\phi_K) = \arccos\frac{-\frac{1}{3}\cdot\frac{1}{\sqrt{6}} - \frac{1}{3}\cdot\frac{1}{\sqrt{6}} + \frac{1}{3}\cdot\frac{2}{\sqrt{6}}}{\sqrt{\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^2}\cdot\sqrt{\frac{1}{6}+\frac{1}{6}+\frac{4}{6}}} = 90^\circ$$

$$\varphi(\phi_I,\phi_L) = \arccos\frac{\frac{1}{3}\cdot\frac{1}{\sqrt{6}} - \frac{1}{3}\cdot\frac{1}{\sqrt{6}} + \frac{1}{3}\cdot 0}{\sqrt{\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^2}\cdot\sqrt{\frac{1}{6}+\frac{1}{6}+0}} = 90^\circ$$

$$\phi(\phi_K,\phi_L) = \arccos\frac{-\frac{1}{\sqrt{6}}\cdot\frac{1}{\sqrt{6}} + \frac{1}{\sqrt{6}}\cdot\frac{1}{\sqrt{6}} + \frac{2}{\sqrt{6}}\cdot 0}{\sqrt{\frac{1}{6}+\frac{1}{6}+\frac{4}{6}}\cdot\sqrt{\frac{1}{6}+\frac{1}{6}+0}} = 90^\circ$$

Interpretation: Only for the *IKL* transform, basis vectors are perpendicular, however the transform is not strictly orthogonal, as the vectors have different Euclidean norms. For YC_rC_b, the axes between chrominance components are approximately perpendicular. For *XYZ*, the angles between the component axes are significantly lower than 90^0, which effects a limitation of the color variations that can be expressed. For none of the transforms, energy preservation is given, such that Euclidean distances between two color points will not be consistent with those in *RGB*.

Problem 4.2

a) If S=0 : $R=G=B=V\cdot A_{\max}$

else :

If $0\le H\le\pi/3$: $R=MAX=V\cdot A_{\max}$; $B=MIN=(1-S)\cdot R$; $G = H\cdot\frac{3}{\pi}\cdot S\cdot R + B$;

If $\pi/3\le H\le\pi$: $G=MAX=V\cdot A_{\max}$; $B-R=\left(H\cdot\frac{3}{\pi}-2\right)\cdot(G-MIN)$;

If $\pi/3\le H\le 2\pi/3$: $B=MIN=(1-S)\cdot G$; $R=\left(2-H\cdot\frac{3}{\pi}\right)\cdot S\cdot G+B$;

If $2\pi/3\le H\le\pi$: $R=MIN=(1-S)\cdot G$; $B=\left(H\cdot\frac{3}{\pi}-2\right)\cdot S\cdot G+R$;

If $\pi\le H\le 5\pi/3$: $B=MAX=V\cdot A_{\max}$; $R-G=\left(H\cdot\frac{3}{\pi}-4\right)\cdot(B-MIN)$;

If $\pi\le H\le 4\pi/3$: $R=MIN=(1-S)\cdot B$; $G=\left(4-H\cdot\frac{3}{\pi}\right)\cdot S\cdot B+R$;

If $4\pi/3\le H\le 5\pi/3$: $G=MIN=(1-S)\cdot B$; $R=\left(H\cdot\frac{3}{\pi}-4\right)\cdot S\cdot B+G$;

If $5\pi/3\le H\le 2\pi$: $R=MAX=V\cdot A_{\max}$; $G=MIN=(1-S)\cdot R$; $B=\left(6-H\cdot\frac{3}{\pi}\right)\cdot S\cdot R+G$;

b) S=1, V=0.5 Yellow : $H=\pi/3 \Rightarrow R=0.5\cdot A_{\max}$; $G=0.5\cdot A_{\max}$; $B=0$

Cyan : $H=\pi \Rightarrow R=0$; $G=0.5\cdot A_{\max}$; $B=0.5\cdot A_{\max}$

Magenta : $H=5\pi/3 \Rightarrow R=0.5\cdot A_{\max}$; $G=0$; $B=0.5\cdot A_{\max}$

S=0.5, V=0.5 Yellow : $H=\pi/3 \Rightarrow R=0.5\cdot A_{\max}$; $G=0.5\cdot A_{\max}$; $B=0.25\cdot A_{\max}$

Cyan : $H=\pi \Rightarrow R=0.25\cdot A_{\max}$; $G=0.5\cdot A_{\max}$; $B=0.5\cdot A_{\max}$

Magenta : $H=5\pi/3 \Rightarrow R=0.5\cdot A_{\max}$; $G=0.25\cdot A_{\max}$; $B=0.5\cdot A_{\max}$

c) According to results of b) or according to the cylinder-shaped color space (Fig. 7.2), the distance from the origin (black) is equal for any color tones of equal *V* and *S* values. For *RGB* according to (7.4), performing normalization by $A_{\max}$:

S=1, V=0.5 : $\mathrm{d}=\sqrt{0.5^2+0.5^2+0^2}=\sqrt{0.5}=\frac{\sqrt{2}}{2}\approx 0.7071$

S=0.5, V=0.5 : $\mathrm{d}=\sqrt{0.5^2+0.5^2+0.25^2}=\sqrt{0.5625}=0.75$

For H,S,V according to (7.7)

S=1, V=0.5 : $\mathrm{d}=\sqrt{\frac{0.5^2+1^2\cdot\left(\cos^2 H+\sin^2 H\right)}{5}}=\sqrt{\frac{1.25}{5}}=\sqrt{0.25}=0.5$

S=0.5, V=0.5 : $\mathrm{d}=\sqrt{\frac{0.5^2+0.5^2\cdot\left(\cos^2 H+\sin^2 H\right)}{5}}=\sqrt{\frac{0.5}{5}}=\sqrt{0.1}\approx 0.316$

The color value of higher saturation has a larger distance from the origin (black) in the HSV color space. This is justified, as a "pure" color tone is perceived (even when having lower brightness) as more different from a black color.

Problem 4.3

a) $D_1=R-I \quad ; \quad D_2=B-I$

b) The basis vectors related to D_1 and D_2 are not orthogonal :

$\frac{2}{3}\cdot\left(-\frac{1}{3}\right)+\left(-\frac{1}{3}\right)\cdot\left(-\frac{1}{3}\right)+\left(-\frac{1}{3}\right)\cdot\frac{2}{3}=-\frac{1}{3}\neq 0$

c) From a) : $R=I+D_1 \quad ; \quad B=I+D_2 \quad ; \quad G=3I-R-B=I-D_1-D_2$

$$\begin{bmatrix}R\\G\\B\end{bmatrix}=\begin{bmatrix}1&1&0\\1&-1&-1\\1&0&1\end{bmatrix}\cdot\begin{bmatrix}I\\D_1\\D_2\end{bmatrix}$$

d) After transformation into ID_1D_2 color space :

$\mathbf{f'}_{A,1}=[0,0,0]^T$; $\mathbf{f'}_{B,1}=[1,0,0]^T$; $\mathbf{f'}_{A,2}=[1/3,2/3,-1/3]^T$; $\mathbf{f'}_{B,2}=[1/3,-1/3,2/3]^T$

RGB : $\mathrm{d}_2(\mathbf{f}_{A,1},\mathbf{f}_{B,1})=\sqrt{1^2+1^2+1^2}=\sqrt{3} \quad ; \quad \mathrm{d}_2(\mathbf{f}_{A,2},\mathbf{f}_{B,2})=\sqrt{1^2+0^2+(-1)^2}=\sqrt{2}$

ID_1D_2 : $\mathrm{d}_2(\mathbf{f}_{A,1},\mathbf{f}_{B,1})=\sqrt{1^2+0^2+0^2}=1; \mathrm{d}_2(\mathbf{f}_{A,2},\mathbf{f}_{B,2})=\sqrt{0^2+1^2+(-1)^2}=\sqrt{2}$

Pair 1 has pure intensity difference, pair 2 pure color difference; it can be concluded that the intensity difference is weighted stronger in case of RGB.

e) $\mu_I=\frac{1}{3}\cdot 5+\frac{1}{3}\cdot 8+\frac{1}{3}\cdot 2=5$; $\sigma_I^{\,2}=\left(\frac{1}{3}\right)^2\cdot 4+\left(\frac{1}{3}\right)^2\cdot 3+\left(\frac{1}{3}\right)^2\cdot 2=1$

Problem 4.4

a) For simplicity, the co-occurence values relating to the last column and row are added to (4.23). Defining

$$\mathbf{C}_\Delta^{(1,0)}=\begin{bmatrix}1&2&1\\3&1&0\\1&0&0\end{bmatrix};\ \mathbf{C}_\Delta^{(0,1)}=\begin{bmatrix}3&2&0\\2&0&1\\0&1&0\end{bmatrix};\ \mathbf{C}_\Delta^{(1,1)}=\begin{bmatrix}3&1&0\\1&1&1\\1&1&0\end{bmatrix}$$

gives the following modifications,

$$\mathbf{C}^{(1,0)}=\begin{bmatrix}3&3&1\\2&3&1\\1&1&1\end{bmatrix}+\mathbf{C}_\Delta^{(1,0)}=\begin{bmatrix}4&5&2\\5&4&1\\2&1&1\end{bmatrix};\ \mathbf{C}^{(0,1)}=\begin{bmatrix}1&4&1\\4&1&2\\1&2&0\end{bmatrix}+\mathbf{C}_\Delta^{(0,1)}=\begin{bmatrix}4&6&1\\6&1&3\\1&3&0\end{bmatrix};$$

$$\mathbf{C}^{(1,1)}=\begin{bmatrix}4&2&1\\2&3&2\\0&2&0\end{bmatrix}+\mathbf{C}_\Delta^{(1,1)}=\begin{bmatrix}7&3&1\\3&4&3\\1&3&0\end{bmatrix}$$

b) The image matrix is a cyclic-shifted (shift by one pixel towards right and bottom) version of (4.21): Therefore, the co-occurrence matrices do not change.

$$\mathbf{S}_{cyc}=\begin{matrix}\begin{bmatrix}0&0&0&1&2\\1&1&0&1&1\\2&2&1&0&0\\1&1&0&2&0\\0&0&1&0&1\end{bmatrix}\begin{bmatrix}0\\1\\2\\1\\0\end{bmatrix}\\\begin{bmatrix}0&0&0&1&2\end{bmatrix}\begin{bmatrix}0\end{bmatrix}\end{matrix}\qquad \mathbf{S}_{cyc,2}=\begin{matrix}\begin{bmatrix}1\end{bmatrix}\begin{bmatrix}0&0&1&0&1\end{bmatrix}\\\begin{bmatrix}2\\1\\0\\0\\1\end{bmatrix}\begin{bmatrix}0&0&0&1&2\\1&1&0&1&1\\2&2&1&0&0\\1&1&0&2&0\\0&0&1&0&1\end{bmatrix}\end{matrix}$$

c) $$\hat{\mathbf{P}}^{(1,0)}=\frac{1}{25}\begin{bmatrix}4&5&2\\5&4&1\\2&1&1\end{bmatrix}\ ;\ \hat{\mathbf{P}}^{(0,1)}=\frac{1}{25}\begin{bmatrix}4&6&1\\6&1&3\\1&3&0\end{bmatrix}\ ;\ \hat{\mathbf{P}}^{(1,1)}=\frac{1}{25}\begin{bmatrix}7&3&1\\3&4&3\\1&3&0\end{bmatrix}$$

$\Rightarrow$ for $\mathbf{P}^{(1,1)}$
according to (4.26) with P=2:

$$\sum_i \sum_j (x_i - x_j)^2 \Pr{}_{i,j}^{(1,1)} = \frac{1}{25}\left(7\cdot 0^2 + 3\cdot 1^2 + 1\cdot 2^2 + 3\cdot 1^2 + 4\cdot 0^2 + 3\cdot 1^2 + 1\cdot 2^2 + 3\cdot 1^2 + 0\cdot 0^2\right) = \frac{20}{25} = 0.8;$$

according to (4.28) with $\log_2$:

$$-\sum_i \sum_j \Pr{}_{i,j}^{(k,l)} \cdot \log_2 \Pr{}_{i,j}^{(1,1)} = 0.514 + 0.367 + 0.186 + 0.367 + 0.423 + 0.367 + 0.186 + 0.367 + 0 = 2.777;$$

according to (4.29):

$$\sum_i \sum_j \left(\Pr{}_{i,j}^{(1,1)}\right)^2 = \frac{1}{25^2}\left(7^2 + 3^2 + 1^2 + 3^2 + 4^2 + 3^2 + 1^2 + 3^2 + 0^2\right) = \frac{103}{625} = 0.1648.$$

Texture 1: Image with equality of entries: $\hat{\mathbf{P}}^{(1,1)} = \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$\Rightarrow$ criterion (4.26) =12/9; criterion (4.28) =$\log_2$ (1/9)=3.1699; criterion (4.29) =1/9

Texture 2: Image with constant gray level (e.g. level 1): $\hat{\mathbf{P}}^{(1,1)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow$ criterion (4.26) =0; criterion (4.28) =0; criterion (4.29) =1

Comparison texture 1: $\mathrm{d} = |0.8 - 0| + |2.777 - 0| + |0.1648 - 1| = 4.4122$

Comparison texture 2: $\mathrm{d} = |0.8 - 1.333| + |2.777 - 3.1699| + |0.1648 - 0.1111| = .9792$

Problem 4.5

a) $$\Pr(0) = \frac{\Pr^{(k,l)}(0,0) + \Pr^{(k,l)}(0,1)}{\Pr^{(k,l)}(0,0) + \Pr^{(k,l)}(0,1) + \Pr^{(k,l)}(1,0) + \Pr^{(k,l)}(1,1)} = 0.5 \quad \Rightarrow \Pr(1) = 1 - \Pr(0) = 0.5$$

b) $m_b = 0.5\cdot 0 + 0.5\cdot 1 = 0.5 \quad ; \quad \sigma_b{}^2 = 0.5\cdot 0^2 + 0.5\cdot 1^2 - m_b{}^2 = 0.25$

c) $$\varphi_{bb}(1,0) = 0\cdot 0\cdot \Pr^{(1,0)}(0,0) + 0\cdot 1\cdot \Pr^{(1,0)}(0,1) + 1\cdot 0\cdot \Pr^{(1,0)}(1,0) + 1\cdot 1\cdot \Pr^{(1,0)}(1,1) = 0.45$$

$$\varphi_{bb}(0,1) = 0\cdot 0\cdot \Pr^{(0,1)}(0,0) + 0\cdot 1\cdot \Pr^{(0,1)}(0,1) + 1\cdot 0\cdot \Pr^{(0,1)}(1,0) + 1\cdot 1\cdot \Pr^{(0,1)}(1,1) = 0.4$$

$$\mu_{bb}(1,0) = \varphi_{bb}(1,0) - m_b{}^2 = 0.2 \quad ; \quad \mu_{bb}(0,1) = \varphi_{bb}(0,1) - m_b{}^2 = 0.15$$

$$\rho_{bb}(1,0) = \frac{\mu_{bb}(1,0)}{\sigma_b{}^2} = 0.8 \quad ; \quad \rho_{bb}(0,1) = \frac{\mu_{bb}(0,1)}{\sigma_b{}^2} = 0.6$$

d) For AR(1) model: A=ρ_{bb}(1,0)=0.8; B=ρ_{bb}(0,1)=0.6; C=0, as AR(1) is zero mean.

Problem 4.6

a) According to (A.24), the frequency coordinate system rotates uniformly with the spatial coordinate system (for alias-free sampled images, this would strictly be true when sampling distances are equal horizontally and vertically):

$$\begin{bmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \cdot \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$$\phi_{ss}(\tilde{f}_1, \tilde{f}_2) = \frac{\sigma_s{}^2(1-\rho^2)}{1 - 2\rho\cos\left(2\pi\sqrt{\left(f_1\cos\alpha + f_2\sin\alpha\right)^2 + \left(f_2\cos\alpha - f_1\sin\alpha\right)^2}\right) + \rho^2}$$

$$= \frac{\sigma_x{}^2(1-\rho^2)}{1 - 2\rho\cos\left(2\pi\sqrt{f_1^2\left(\cos^2\alpha + \sin^2\alpha\right) + f_2^2\left(\cos^2\alpha + \sin^2\alpha\right)}\right) + \rho^2} = \phi_{ss}(f_1, f_2).$$

As the autocovariance statistics of the isotropic model are invariant against rotations, the power spectrum does not change.

b) The modification of frequency coordinates in case of pure scaling results analogously with (A.14):

$$\begin{bmatrix} \tilde{n}_1 \\ \tilde{n}_2 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \cdot \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} \Rightarrow \begin{bmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$$\begin{bmatrix} \tilde{n}_1 \\ \tilde{n}_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} \Rightarrow \begin{bmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \cdot \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$\Rightarrow$ for scaling 0.5: $$\phi_{ss}(\tilde{f}_1, \tilde{f}_2) = \frac{\sigma_s{}^2(1-\rho^2)}{1 - 2\rho\cos\left(2\pi\sqrt{4f_1^2 + 4f_2^2}\right) + \rho^2} \cdot \left\| \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \right\|^2 = \frac{\frac{1}{16}\sigma_s{}^2(1-\rho^2)}{1 - 2\rho\cos\left(4\pi\sqrt{f_1^2 + f_2^2}\right) + \rho^2} = \frac{1}{16}\phi_{ss}\left(2f_1, 2f_2\right)$$

$$\Rightarrow \text{for scaling 2: } \phi_{ss}(\tilde{f}_1,\tilde{f}_2)=\frac{\sigma_s^2(1-\rho^2)}{1-2\rho\cos\left(2\pi\sqrt{0.25f_1^2+0.25f_2^2}\right)+\rho^2}\cdot\left\|\begin{bmatrix}2&0\\0&2\end{bmatrix}\right\|^2=\frac{16\sigma_s^2(1-\rho^2)}{1-2\rho\cos\left(\pi\sqrt{f_1^2+f_2^2}\right)+\rho^2}=16\phi_{ss}\left(0.5f_1,0.5f_2\right)$$

Problem 4.7

a)

$$\mathbf{G}_{\mathrm{h}}=\frac{1}{3}\begin{bmatrix}X&X&X&X&X&X\\X&0&5&5&5&X\\X&5&10&10&5&X\\X&5&5&5&0&X\\X&10&5&0&0&X\\X&X&X&X&X&X\end{bmatrix}\qquad \mathbf{G}_{\mathrm{v}}=\frac{1}{3}\begin{bmatrix}X&X&X&X&X&X\\X&0&5&15&10&X\\X&5&10&10&5&X\\X&10&15&5&0&X\\X&10&10&0&0&X\\X&X&X&X&X&X\end{bmatrix}$$

$$\mathbf{G}_{\mathrm{d}^+}=\frac{1}{3}\begin{bmatrix}X&X&X&X&X&X\\X&0&0&5&5&X\\X&0&0&0&0&X\\X&5&5&0&0&X\\X&0&5&0&0&X\\X&X&X&X&X&X\end{bmatrix}\qquad \mathbf{G}_{\mathrm{d}^-}=\frac{1}{3}\begin{bmatrix}X&X&X&X&X&X\\X&0&5&15&10&X\\X&5&15&15&5&X\\X&10&15&5&0&X\\X&15&10&0&0&X\\X&X&X&X&X&X\end{bmatrix}$$

b)

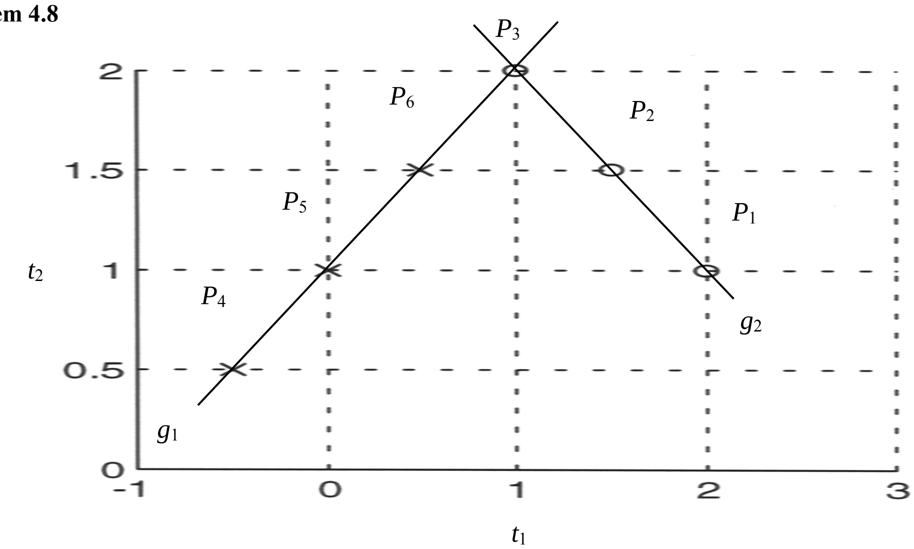
$$\mathbf{G}=\frac{1}{3}\begin{bmatrix}X&X&X&X&X&X\\X&0&5&15&10&X\\X&5&15&15&5&X\\X&10&15&5&0&X\\X&15&10&0&0&X\\X&X&X&X&X&X\end{bmatrix}\quad \text{with } 10/3<\Theta\le 15/3: \quad \mathbf{B}=\begin{bmatrix}X&X&X&X&X&X\\X&0&0&\mathbf{1}&0&X\\X&0&\mathbf{1}&\mathbf{1}&0&X\\X&0&\mathbf{1}&0&0&X\\X&\mathbf{1}&0&0&0&X\\X&X&X&X&X&X\end{bmatrix}$$

c)

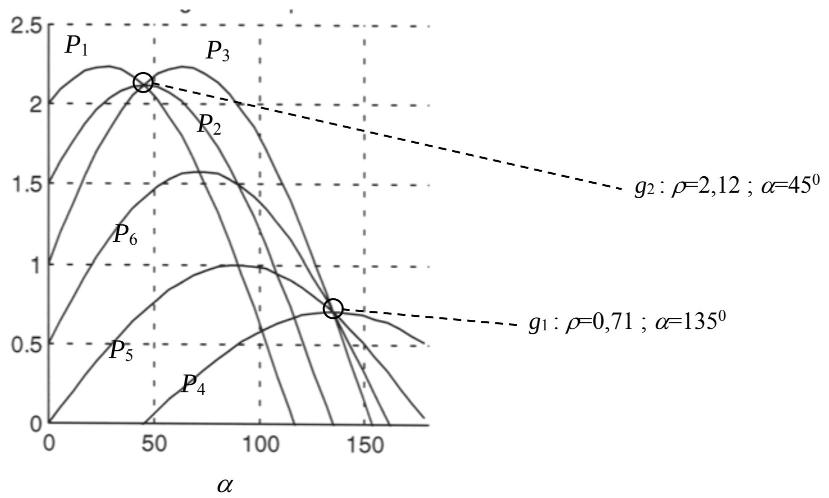
$$\mathbf{G}=\frac{1}{4}\begin{bmatrix}X&X&X&X&X&X\\X&0&0&-10&5&X\\X&0&-10&10&0&X\\X&-5&10&0&0&X\\X&-10&5&0&0&X\\X&X&X&X&X&X\end{bmatrix}\quad \mathbf{B}=\begin{bmatrix}X&X&X&X&X&X\\X&0&0&\mathbf{1}&0&X\\X&0&\mathbf{1}&\mathbf{1}&0&X\\X&0&\mathbf{1}&0&0&X\\X&\mathbf{1}&0&0&0&X\\X&X&X&X&X&X\end{bmatrix}$$

Problem 4.8

a)



b)



Example for computation of the intersection P_1/P_2:

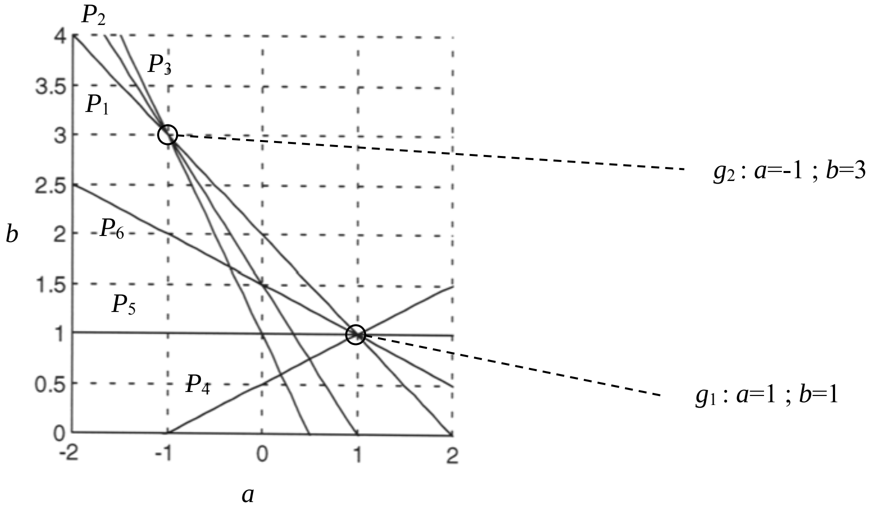
$P_1 : \rho = 2\cos\alpha + \sin\alpha \quad ; \quad P_2 : \rho = 1,5\cos\alpha + 1,5\sin\alpha$

$\Rightarrow 2\cos\alpha + \sin\alpha = 1,5\cos\alpha + 1,5\sin\alpha$

$\Rightarrow \cos\alpha = \sin\alpha \Rightarrow \alpha = \pm\frac{\pi}{4} \Rightarrow \rho = \pm 2\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \pm\frac{3}{2}\sqrt{2}$

The negative value of ρ cannot exist, such that a unique solution with positive α exists.

c) $t_2 = at_1 + b \Rightarrow b = -t_1 a + t_2$



d) The starting and ending points of lines are described by the respective intersecting graphs in the Hough space having highest and lowest slopes (for the case of Cartesian Hough space) or highest and lowest angles (for the case of polar Hough space).

Problem 4.9

a)

n_3=	0	1	2	3	4	5	6	7
$n_1(n_3)$	**1**	**2**	**3**	**4**	**3**	**2**	**1**	**1**
$n_2(n_3)$	**1**	**1**	**1**	**2**	**3**	**4**	**3**	**2**

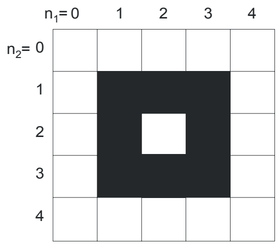
n_3=	0	1	2	3	4	5	6	7
$n_1(n_3)$	**1**	**2**	**3**	**3**	**3**	**2**	**1**	**1**
$n_2(n_3)$	**1**	**1**	**1**	**2**	**3**	**3**	**3**	**2**

b) Original contour:

$L = 4\left(\sqrt{2}+1\right) \approx 9,64$; a_1=a_2=4; A=11; ξ_K=11/16; ξ_F=16

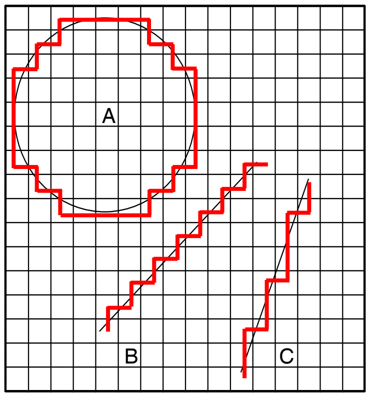
Filtered contour (see Figure below):

L=8; a_1=a_2=3; A=9; ξ_K=1; ξ_F=16



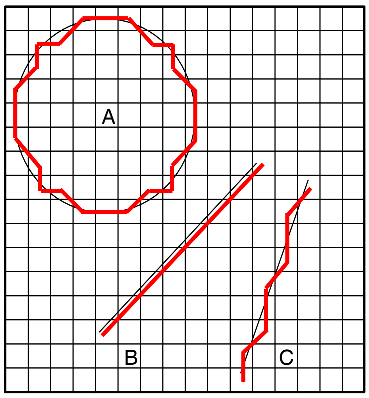
Problem 4.10
Lengths of continuous contours:
Shape A : Circle with radius =4, ⇒L_{orig}=perimeter=2π·4=25.13 ;
Shape B : $L_{orig} = \sqrt{7^2 + 7^2} \approx 9.9$; Shape C : $L_{orig} = \sqrt{3^2 + 8^2} \approx 8.54$

a)



The discrete contours are designed by connecting the centers of coordinate cells, such that the area between the continuous contour and the discrete approximation thereof becomes minimum. In the case of 4-neighbor ($\mathcal{N}_1^{(1)}$) connections, each connecting element has the length of 1. Shape A : L=32 ; Shape B : L=14 ; Shape C : L=12

b)



In the case of the 8-neighbor system ($\mathcal{N}_2^{(2)}$), horizontal and vertical connecting lines have a length of 1, while diagonal connecting lines have a length of $\sqrt{2}$.
Shape A: $L = 16 + 8 \cdot \sqrt{2} \approx 27.31$; B: $L = 7 \cdot \sqrt{2} \approx 9.9$; C: $L = 5 + 3 \cdot \sqrt{2} \approx 9.24$
In general, the approximation is better by using this system; the discrete contour is typically ≥L_{orig} (unless majority of vertices is inside the original contour), which would usual,y not be the case when the constraint stated above (minimization of area between contour and approximation) is observed.

Problem 4.11
1. Around each skeleton sample, perform N subsequent morphological dilation operations where N is equal to the distance value. The shape of the structure element has to be selected depending on the meaning of the distance (e.g., an $\mathcal{N}_1$ shape is used when the distance is only measured horizontally/vertically).
2. The reconstructed shape is the unification of the dilated results around all skeleton values. For implementation, this can be realized by initializing the reconstruction array by zero values, dilate the skeleton samples seqentially and change a position of the reconstruction array into one when any of the dilated results has a logical one at this position.

Problem 4.12

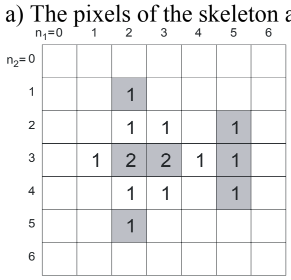
$$\rho^{(2,0)} = \frac{\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}(n_1-\overline{n}_1)^2 s(n_1,n_2)}{\mu^{(0,0)}} = \frac{\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} n_1^{\,2} s(n_1,n_2)}{\mu^{(0,0)}} - 2\overline{n}_1 \frac{\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} n_1 s(n_1,n_2)}{\mu^{(0,0)}} + \overline{n}_1^{\,2} \frac{\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} s(n_1,n_2)}{\mu^{(0,0)}}$$

$$= \frac{\mu^{(2,0)}}{\mu^{(0,0)}} - 2\overline{n}_1 \frac{\mu^{(1,0)}}{\mu^{(0,0)}} + \overline{n}_1^{\,2} = \frac{\mu^{(2,0)}}{\mu^{(0,0)}} - \overline{n}_1^{\,2}$$

Similarly, $\rho^{(0,2)} = \dfrac{\mu^{(0,2)}}{\mu^{(0,0)}} - \overline{n}_2^{\,2}$.

Problem 4.13

a) The pixels of the skeleton are shown by gray shading.



b)

n_1 | n_2 =	0	1	2	3	4	5	6
$\Pi_v(n_1)$ =	0	1	5	3	1	3	0
$\Pi_h(n_2)$ =	0	1	3	5	3	1	0

c) The computation can be performed from the projection profiles, similar to the computation of statistical moments from histogram values (in principle, the projection profiles are histograms of coordinate distributions). Moments of orders 0 and 1, and subsequently the centroid coordinates of the shape are determined as follows:

$$\overline{n}_1 = \frac{\mu^{(1,0)}}{\mu^{(0,0)}} = \frac{\sum_{n_1=0}^{N_1-1} n_1 \Pi_v(n_1)}{\sum_{n_1=0}^{N_1-1} \Pi_v(n_1)} = \frac{0\cdot 0+1\cdot 1+2\cdot 5+3\cdot 3+4\cdot 1+5\cdot 3+6\cdot 0}{13} = \frac{39}{13} = 3$$

$$\overline{n}_2 = \frac{\mu^{(0,1)}}{\mu^{(0,0)}} = \frac{\sum_{n_2=0}^{N_2-1} n_2 \Pi_h(n_2)}{\sum_{n_2=0}^{N_2-1} \Pi_h(n_2)} = \frac{0\cdot 0+1\cdot 1+2\cdot 3+3\cdot 5+4\cdot 3+5\cdot 1+6\cdot 0}{13} = \frac{39}{13} = 3$$

d) The computation of moments $\mu^{(2,0)}$ and $\mu^{(0,2)}$ is performed similar to c):

$$\mu^{(2,0)} = \sum_{n_1=0}^{N_1-1} n_1^{\,2}\Pi_v(n_1) = 0^2\cdot 0+1^2\cdot 1+2^2\cdot 5+3^2\cdot 3+4^2\cdot 1+5^2\cdot 3+6^2\cdot 0 = 139$$

$$\mu^{(0,2)} = \sum_{n_2=0}^{N_2-1} n_2^{\,2}\Pi_h(n_2) = 0^2\cdot 0+1^2\cdot 1+2^2\cdot 3+3^2\cdot 5+4^2\cdot 3+5^2\cdot 1+6^2\cdot 0 = 131$$

Using the results from Problems 4.11 and 4.12c gives:

$$\rho^{(2,0)} = \frac{\mu^{(2,0)}}{\mu^{(0,0)}} - \overline{n}_1^{\,2} = \frac{139}{13} - 3^2 = \frac{22}{13} \approx 1.69$$

$$\rho^{(0,2)} = \frac{\mu^{(0,2)}}{\mu^{(0,0)}} - \overline{n}_2^{\,2} = \frac{131}{13} - 3^2 = \frac{14}{13} \approx 1.08$$

The moment $\mu^{(1,1)}$ can only be computed directly from the image (it plays a similar role as a covariance, which cannot be computed from the type of first-order distributions as the projection profiles are). The value computations for cases $s(n_1,n_2)\neq 0$ are performed in row-wise sequence in the subsequent formula:

$$\mu^{(1,1)} = \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1} n_1 n_2 s(n_1,n_2) = \sum_{n_2=0}^{N_2-1} n_2 \sum_{n_1=0}^{N_1-1} n_1 s(n_1,n_2)$$

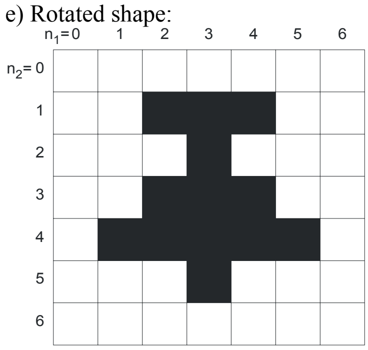
$$= 1\cdot 2 + 2\cdot\left(2+3+5\right) + 3\cdot\left(1+2+3+4+5\right) + 4\cdot\left(2+3+5\right) + 5\cdot 2 = 117$$

With (4.131) and the results of c) this gives $\rho^{(1,1)} = \dfrac{\mu^{(1,1)}}{\mu^{(0,0)}} - \overline{n}_1\overline{n}_2 = \dfrac{117}{13} - 3\cdot 3 = 0 \Rightarrow \mathbf{\Gamma} = \dfrac{1}{13}\begin{bmatrix} 22 & 0 \\ 0 & 14 \end{bmatrix}$

$$u_1 = \rho^{(2,0)} + \rho^{(0,2)} = \frac{22+14}{13} = \frac{36}{13}$$

$$u_2 = \sqrt{\left(\rho^{(2,0)} - \rho^{(0,2)}\right)^2 + 4\left(\rho^{(1,1)}\right)^2} = \frac{\sqrt{(22-14)^2}}{13} = \frac{8}{13}$$

e) Rotated shape:



$$\tilde{\rho}^{(2,0)} = \rho^{(0,2)} \quad ; \quad \tilde{\rho}^{(0,2)} = \rho^{(2,0)}$$

$$\tilde{\mu}^{(1,1)} = 1\cdot(2+3+4) + 2\cdot 3 + 3\cdot(2+3+4) + 4\cdot(1+2+3+4+5) + 5\cdot 3 = 117 \quad \Rightarrow \quad \tilde{\rho}^{(1,1)} = 0$$

$$\tilde{\mathbf{\Gamma}} = \frac{1}{13}\begin{bmatrix} 14 & 0 \\ 0 & 22 \end{bmatrix}$$

$$\tilde{u}_1 = \tilde{\rho}^{(2,0)} + \tilde{\rho}^{(0,2)} = \frac{14+22}{13} = \frac{36}{13} = u_1 \;;\; \tilde{u}_2 = \sqrt{\left(\tilde{\rho}^{(2,0)} - \tilde{\rho}^{(0,2)}\right)^2 + 4\left(\tilde{\rho}^{(1,1)}\right)^2} = \frac{\sqrt{(14-22)^2}}{13} = \frac{8}{13} = u_2$$

Problem 4.14

a) Nine positions.

b) Shifted Versions $\mathbf{S}^{(k_1,k_2)}$ from $\mathbf{S}(n_3-1)$ are as follows; this shift is performed such that further processing can be made between frames $\mathbf{S}(n_3)$ and $\mathbf{S}(n_3-1)$ at same coordinate positions:

$$\mathbf{S}^{(1,1)} = \begin{bmatrix} 5&5&9&9&9&X \\ 5&5&9&9&9&X \\ 5&9&9&9&9&X \\ 9&9&9&9&9&X \\ 5&5&5&5&5&X \\ X&X&X&X&X&X \end{bmatrix} \quad \mathbf{S}^{(0,1)} = \begin{bmatrix} 5&5&5&9&9&9 \\ 5&5&5&9&9&9 \\ 5&5&9&9&9&9 \\ 5&9&9&9&9&9 \\ 5&5&5&5&5&5 \\ X&X&X&X&X&X \end{bmatrix} \quad \mathbf{S}^{(-1,1)} = \begin{bmatrix} X&5&5&5&9&9 \\ X&5&5&5&9&9 \\ X&5&5&9&9&9 \\ X&5&9&9&9&9 \\ X&5&5&5&5&5 \\ X&X&X&X&X&X \end{bmatrix}$$

$$\mathbf{S}^{(1,0)} = \begin{bmatrix} 5&5&9&9&9&X \\ 5&5&9&9&9&X \\ 5&5&9&9&9&X \\ 5&9&9&9&9&X \\ 9&9&9&9&9&X \\ 5&5&5&5&5&X \end{bmatrix} \quad \mathbf{S}^{(0,0)} = \begin{bmatrix} 5&5&5&9&9&9 \\ 5&5&5&9&9&9 \\ 5&5&5&9&9&9 \\ 5&5&9&9&9&9 \\ 5&9&9&9&9&9 \\ 5&5&5&5&5&5 \end{bmatrix} \quad \mathbf{S}^{(-1,0)} = \begin{bmatrix} X&5&5&5&9&9 \\ X&5&5&5&9&9 \\ X&5&5&5&9&9 \\ X&5&5&9&9&9 \\ X&5&9&9&9&9 \\ X&5&5&5&5&5 \end{bmatrix}$$

$$\mathbf{S}^{(1,-1)} = \begin{bmatrix} X&X&X&X&X&X \\ 5&5&9&9&9&X \\ 5&5&9&9&9&X \\ 5&5&9&9&9&X \\ 5&9&9&9&9&X \\ 9&9&9&9&9&X \end{bmatrix} \quad \mathbf{S}^{(0,-1)} = \begin{bmatrix} X&X&X&X&X&X \\ 5&5&5&9&9&9 \\ 5&5&5&9&9&9 \\ 5&5&5&9&9&9 \\ 5&5&9&9&9&9 \\ 5&9&9&9&9&9 \end{bmatrix} \quad \mathbf{S}^{(-1,-1)} = \begin{bmatrix} X&X&X&X&X&X \\ X&5&5&5&9&9 \\ X&5&5&5&9&9 \\ X&5&5&5&9&9 \\ X&5&5&9&9&9 \\ X&5&9&9&9&9 \end{bmatrix}$$

The related motion-compensated prediction error blocks $\mathbf{E}^{(k,l)} = \mathbf{S}(n_3) - \mathbf{S}^{(k_1,k_2)}$ are

$$\mathbf{E}^{(1,1)} = \begin{bmatrix} X&X&X&X&X&X \\ X&0&-4&-2&0&X \\ X&-4&-2&0&0&X \\ X&-2&0&0&0&X \\ X&0&0&0&0&X \\ X&X&X&X&X&X \end{bmatrix} \quad \mathbf{E}^{(0,1)} = \begin{bmatrix} X&X&X&X&X&X \\ X&0&0&-2&0&X \\ X&0&-2&0&0&X \\ X&-2&0&0&0&X \\ X&0&0&0&0&X \\ X&X&X&X&X&X \end{bmatrix} \quad \mathbf{E}^{(-1,1)} = \begin{bmatrix} X&X&X&X&X&X \\ X&0&0&-2&0&X \\ X&0&-2&0&0&X \\ X&-2&0&0&0&X \\ X&0&0&0&0&X \\ X&X&X&X&X&X \end{bmatrix}$$

$$\mathbf{E}^{(1,0)} = \begin{bmatrix} X&X&X&X&X&X \\ X&0&-4&-2&0&X \\ X&0&-2&0&0&X \\ X&-2&0&0&0&X \\ X&-4&-4&-4&-4&X \\ X&X&X&X&X&X \end{bmatrix} \quad \mathbf{E}^{(0,0)} = \begin{bmatrix} X&X&X&X&X&X \\ X&0&0&-2&0&X \\ X&0&2&0&0&X \\ X&2&0&0&0&X \\ X&-4&-4&-4&-4&X \\ X&X&X&X&X&X \end{bmatrix} \quad \mathbf{E}^{(-1,0)} = \begin{bmatrix} X&X&X&X&X&X \\ X&0&0&2&0&X \\ X&0&2&4&0&X \\ X&2&4&0&0&X \\ X&0&-4&-4&-4&X \\ X&X&X&X&X&X \end{bmatrix}$$

$$\mathbf{E}^{(1,-1)}=\begin{bmatrix} X & X & X & X & X & X \\ X & 0 & -4 & -2 & 0 & X \\ X & 0 & -2 & 0 & 0 & X \\ X & 2 & 0 & 0 & 0 & X \\ X & -4 & -4 & -4 & -4 & X \\ X & X & X & X & X & X \end{bmatrix} \quad \mathbf{E}^{(0,-1)}=\begin{bmatrix} X & X & X & X & X & X \\ X & 0 & 0 & -2 & 0 & X \\ X & 0 & 2 & 0 & 0 & X \\ X & 2 & 4 & 0 & 0 & X \\ X & 0 & -4 & -4 & -4 & X \\ X & X & X & X & X & X \end{bmatrix} \quad \mathbf{E}^{(-1,-1)}=\begin{bmatrix} X & X & X & X & X & X \\ X & 0 & 0 & 2 & 0 & X \\ X & 0 & 2 & 4 & 0 & X \\ X & 2 & 4 & 4 & 0 & X \\ X & 0 & 0 & -4 & -4 & X \\ X & X & X & X & X & X \end{bmatrix}$$

Usage of the minimum squared error $\|\mathbf{E}\|^2$ as cost function indicates two equally optimum results (both with minimum cost =12) for $(k_1,k_2)=(0,1)$ and $(k_1,k_2)=(-1,1)$:

$$\mathbf{K}^{(\sigma_e{}^2)}=\frac{1}{16}\begin{bmatrix} 44 & 12 & 12 \\ 92 & 76 & 92 \\ 92 & 76 & 92 \end{bmatrix}$$

Pixel-wise multiplication $\mathrm{M}^{(k_1,k_2)}=\mathbf{S}(n_3)\circ\mathbf{S}^{(k_1,k_2)}$ is performed to compute the cross correlation:

$$\mathbf{M}^{(1,1)}=\begin{bmatrix} X & X & X & X & X & X \\ X & 25 & 45 & 63 & 81 & X \\ X & 45 & 63 & 81 & 81 & X \\ X & 63 & 81 & 81 & 81 & X \\ X & 25 & 25 & 25 & 25 & X \\ X & X & X & X & X & X \end{bmatrix} \quad \mathbf{M}^{(0,1)}=\begin{bmatrix} X & X & X & X & X & X \\ X & 25 & 25 & 63 & 81 & X \\ X & 25 & 63 & 81 & 81 & X \\ X & 63 & 81 & 81 & 81 & X \\ X & 25 & 25 & 25 & 25 & X \\ X & X & X & X & X & X \end{bmatrix} \quad \mathbf{M}^{(-1,1)}=\begin{bmatrix} X & X & X & X & X & X \\ X & 25 & 25 & 35 & 81 & X \\ X & 25 & 35 & 81 & 81 & X \\ X & 35 & 81 & 81 & 81 & X \\ X & 25 & 25 & 25 & 25 & X \\ X & X & X & X & X & X \end{bmatrix}$$

$$\mathbf{M}^{(1,0)}=\begin{bmatrix} X & X & X & X & X & X \\ X & 25 & 45 & 63 & 81 & X \\ X & 25 & 63 & 81 & 81 & X \\ X & 63 & 81 & 81 & 81 & X \\ X & 45 & 45 & 45 & 45 & X \\ X & X & X & X & X & X \end{bmatrix} \quad \mathbf{M}^{(0,0)}=\begin{bmatrix} X & X & X & X & X & X \\ X & 25 & 25 & 63 & 81 & X \\ X & 25 & 35 & 81 & 81 & X \\ X & 35 & 81 & 81 & 81 & X \\ X & 45 & 45 & 45 & 45 & X \\ X & X & X & X & X & X \end{bmatrix} \quad \mathbf{M}^{(-1,0)}=\begin{bmatrix} X & X & X & X & X & X \\ X & 25 & 25 & 35 & 81 & X \\ X & 25 & 35 & 45 & 81 & X \\ X & 35 & 45 & 81 & 81 & X \\ X & 25 & 45 & 45 & 45 & X \\ X & X & X & X & X & X \end{bmatrix}$$

$$\mathbf{M}^{(1,-1)}=\begin{bmatrix} X & X & X & X & X & X \\ X & 25 & 45 & 63 & 81 & X \\ X & 25 & 63 & 81 & 81 & X \\ X & 35 & 81 & 81 & 81 & X \\ X & 45 & 45 & 45 & 45 & X \\ X & X & X & X & X & X \end{bmatrix} \quad \mathbf{M}^{(0,-1)}=\begin{bmatrix} X & X & X & X & X & X \\ X & 25 & 25 & 63 & 81 & X \\ X & 25 & 35 & 81 & 81 & X \\ X & 35 & 45 & 81 & 81 & X \\ X & 25 & 45 & 45 & 45 & X \\ X & X & X & X & X & X \end{bmatrix} \quad \mathbf{M}^{(-1,-1)}=\begin{bmatrix} X & X & X & X & X & X \\ X & 25 & 25 & 35 & 81 & X \\ X & 25 & 35 & 45 & 81 & X \\ X & 35 & 45 & 45 & 81 & X \\ X & 25 & 25 & 81 & 81 & X \\ X & X & X & X & X & X \end{bmatrix}$$

The values of cross correlation are obtained by adding the values within each of the matrices. The highest value (950/16) corresponds to $(k_1,k_2)=(1,0)$, which is however less clear differentiated from the other values here:

$$\mathbf{K}^{(\varphi_{sg})}=\frac{1}{16}\begin{bmatrix} 890 & 769 & 685 \\ 950 & 874 & 754 \\ 922 & 818 & 770 \end{bmatrix}$$

For a more evident result, normalized cross covariance should be used. (this may be exercised as an additional problem)

2. The horizontal shift by half a sample using linear interpolation is equivalent with the following averaging operation:

$$\mathbf{S}^{(-\frac{1}{2},1)}=\mathbf{S}^{(0,1)}+\mathbf{S}^{(-1,1)}=\frac{1}{2}\left[\begin{bmatrix} 5 & 5 & 5 & 9 & 9 & 9 \\ 5 & 5 & 5 & 9 & 9 & 9 \\ 5 & 5 & 9 & 9 & 9 & 9 \\ 5 & 9 & 9 & 9 & 9 & 9 \\ 5 & 5 & 5 & 5 & 5 & 5 \\ X & X & X & X & X & X \end{bmatrix}+\begin{bmatrix} X & 5 & 5 & 5 & 9 & 9 \\ X & 5 & 5 & 5 & 9 & 9 \\ X & 5 & 5 & 9 & 9 & 9 \\ X & 5 & 9 & 9 & 9 & 9 \\ X & 5 & 5 & 5 & 5 & 5 \\ X & X & X & X & X & X \end{bmatrix}\right]=\begin{bmatrix} X & X & X & X & X & X \\ X & 5 & 5 & 7 & 9 & X \\ X & 5 & 7 & 9 & 9 & X \\ X & 7 & 9 & 9 & 9 & X \\ X & 5 & 5 & 5 & 5 & X \\ X & X & X & X & X & X \end{bmatrix}$$

Within the current block, the resultant matrix is exactly equal to the matrix $\mathbf{S}(n_3)$. The cost function using minimum squared difference will give the value of zero.

Problem 4.15

$$\tilde{t}_1=a_1t_1-a_2t_2+d_1 \;\Rightarrow\; \tilde{t}_1-t_1=\left(a_1-1\right)t_1-a_2t_2+d_1 \;\Rightarrow\; k_1(n_1,n_2)T_1=\left(a_1-1\right)n_1T_1-a_2n_2T_2+d_1$$

$$\tilde{t}_2=a_2t_1+a_1t_2+d_2 \;\Rightarrow\; \tilde{t}_2-t_2=a_2t_1+\left(a_1-1\right)t_2+d_2 \;\Rightarrow\; k_2(n_1,n_2)T_2=a_2n_1T_1+\left(a_1-1\right)n_2T_2+d_2$$

Normalization by sampling intervals:

$$k_1(n_1,n_2)=(a_1-1)n_1-a_2\frac{T_2}{T_1}n_2+\frac{d_1}{T_1} \quad ; \quad k_2(n_1,n_2)=a_2\frac{T_1}{T_2}n_1+(a_1-1)n_2+\frac{d_2}{T_2}$$

Substitution into optical flow equation:

$$\left[(a_1-1)n_1-a_2\frac{T_2}{T_1}n_2+\frac{d_1}{T_1}\right]s_1(n_1,n_2)+\left[a_2\frac{T_1}{T_2}n_1+(a_1-1)n_2+\frac{d_2}{T_2}\right]s_2(n_1,n_2)=-s_3(n_1,n_2)$$

The resulting over-determined equation system is

$$\begin{bmatrix} n_1 s_1(1)+n_2 s_2(1) & n_1 \frac{T_1}{T_2} s_2(1) - n_2 \frac{T_2}{T_1} s_1(1) & \frac{1}{T_1} s_1(1) & \frac{1}{T_2} s_2(1) \\ n_1 s_1(2)+n_2 s_2(2) & n_1 \frac{T_1}{T_2} s_2(2) - n_2 \frac{T_2}{T_1} s_1(2) & \frac{1}{T_1} s_1(2) & \frac{1}{T_2} s_2(2) \\ \vdots & \vdots & \vdots & \vdots \\ n_1 s_1(P)+n_2 s_2(P) & n_1 \frac{T_1}{T_2} s_2(P) - n_2 \frac{T_2}{T_1} s_1(P) & \frac{1}{T_1} s_1(P) & \frac{1}{T_2} s_2(P) \end{bmatrix} \cdot \begin{bmatrix} a_1-1 \\ a_2 \\ d_1 \\ d_2 \end{bmatrix} = - \begin{bmatrix} s_3(1) \\ s_3(2) \\ \vdots \\ s_3(P) \end{bmatrix}$$

Additional Problem 4.16 (not in book)

Zoom, rotation and translation of a camera shall be estimated by a 4-parameter model (see problem 4.15). The sampling distance is $T_1=T_2=1$ mm. The following discrete motion vectors (pixel shift) are available for two positions in the image: $k_1(0,0)=2$; $k_2(0,0)=3$; $k_1(1,1)=1$; $k_2(1,1)=4$. The optical axis intersects the image plane at (0,0).

a) Determine equations for $k_1(n_1,n_2)$ and $k_2(n_1,n_2)$ in dependency of a_1, a_2, t_1 and t_2. Then, compute the latter 4 parameters.

b) By which speed was the camera moved horizontally/vertically , if the temporal distance between the frames is 40 ms?

c) Would the parameters computed in a) change, if the optical axis would intersect the image plane at (1,1)?

d) Now, $a_1=1$ and $a_2=-1$, and assume it is known that the camera is rotated around the optical axis. Determine the rotation angle θ and the scaling factor Θ.

e) What would be the effect of $\Theta<0$?

f) Determine the parameters for cases where the image is mirrored about the i) horizontal and ii) vertical coordinate axes.

Solution:

a)
$$\tilde{t}_1 = a_1 t_1 - a_2 t_2 + d_1 \Rightarrow \tilde{t}_1 - t_1 = (a_1-1)t_1 - a_2 t_2 + d_1 \Rightarrow k_1(n_1,n_2)T_1 = (a_1-1)n_1 T_1 - a_2 n_2 T_2 + d_1$$
$$\tilde{t}_2 = a_2 t_1 + a_1 t_2 + d_2 \Rightarrow \tilde{t}_2 - t_2 = a_2 t_1 + (a_1-1)t_2 + d_2 \Rightarrow k_2(n_1,n_2)T_2 = a_2 n_1 T_1 + (a_1-1)n_2 T_2 + d_2$$
$$\Rightarrow k_1(n_1,n_2) = (a_1-1)n_1 - a_2 n_2 + \frac{d_1}{mm}; \quad k_2(n_1,n_2) = a_2 n_1 + (a_1-1)n_2 + \frac{d_2}{mm}$$

$$k_1(0,0) = 2 = (a_1-1)\cdot 0 - a_2 \cdot 0 + \frac{d_1}{\text{mm}} \Rightarrow d_1 = 2\text{ mm}; \quad k_2(0,0) = 3 = a_2 \cdot 0 + (a_1-1)\cdot 0 + \frac{d_2}{\text{mm}} \Rightarrow d_2 = 3\text{ mm}$$
$$k_2(1,1) = 1 = (a_1-1)\cdot 1 - a_2\cdot 1 + 2 \Rightarrow a_1 - a_2 = 0; \quad k_2(1,1) = 4 = a_2\cdot 1 + (a_1-1)\cdot 1 + 3 \Rightarrow a_1 + a_2 = 2 \Rightarrow a_1 = a_2 = 1$$
$$v_1 = \frac{2\text{ mm}}{40\text{ ms}} = 5\frac{\text{cm}}{\text{s}}; \quad v_2 = \frac{3\text{ mm}}{40\text{ ms}} = 7.5\frac{\text{cm}}{\text{s}}$$

b) Yes, as only the centers of rotation and scaling are changed.

c)
$$a_1 = \Theta\cdot\cos\theta = 1; \quad a_2 = \Theta\cdot\sin\theta = -1$$
$$\Rightarrow \cos\theta = -\sin\theta; \quad \cos^2\theta + \sin^2\theta = 1 = 2\cos^2\theta \Rightarrow \cos\theta \overset{\Theta>0}{=} \frac{\sqrt{2}}{2} \Rightarrow \theta \overset{\sin\theta=-\frac{\sqrt{2}}{2}}{=} -45^0 \Rightarrow \Theta = \sqrt{2}$$

d) Change of sign in cos and sin functions equals rotation by 180^0.

e)
Mirror by horizontal axis: $\begin{bmatrix} \tilde{t}_1 \\ \tilde{t}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$

Mirror by vertical axis: $\begin{bmatrix} \tilde{t}_1 \\ \tilde{t}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$

Additional Problem 4.17 (not in book)

The horizontal motion of an image line shall be estimated. The following samples are available from two subsequent images, where the horizontal motion shift k shall be estimated in one row n_2 for the marked area:

$$\mathbf{s}(n_2,n_3) = \left[\boxed{4 \quad 3 \quad 4 \quad 3} \; 2 \quad 4\right]^{\mathrm{T}} \qquad \mathbf{s}(n_2,n_3-1) = \begin{bmatrix} 3 & 3 & 5 & 1 & 3 & 5 \end{bmatrix}^{\mathrm{T}}$$

a) Determine the discrete optical flow equation for the computation of k_1 and the over-determined equation system (matrix formulation representing 4 equations).

b) Determine k_1 by computing the pseudo inverse.

c) Compute the values for half-sample positions in $s(n_3-1)$ by linear interpolation (i.e. averaging adjacent pixels).

d) Compute the displaced picture differences $s(n_1,n_2,n_3) - s(n_1-k_1,n_2,n_3-1)$ using the result of b) for the four highlighted values. Would an improvement of motion estimation be expected from one more iteration?

Solution

a) $\mathbf{s}_1(n_2,n_3) = \begin{bmatrix} -1 & 1 & -1 & -1 \end{bmatrix}^{\mathrm{T}}$; $\mathbf{s}_3(n_2,n_3) = \begin{bmatrix} 0 & -1 & 2 & -1 \end{bmatrix}^{\mathrm{T}}$

Note: $\mathbf{s}_1$ corresponds to $\mathbf{S}$ in (4.165), $\mathbf{s}_3$ corresponds to $-\mathbf{s}$

$$k_1 s_1(n_2,n_3) = -s_3(n_2,n_3) \Rightarrow \underbrace{\begin{bmatrix}-1\\1\\-1\\-1\end{bmatrix}}_{\mathbf{s}} k_1 = \underbrace{\begin{bmatrix}0\\1\\-2\\1\end{bmatrix}}_{\mathbf{s}}$$

b)

$$\mathbf{S}^{\mathrm{p}} = \left[\underbrace{\begin{bmatrix}-1 & 1 & -1 & -1\end{bmatrix}}_{\mathbf{s}^{\mathrm{T}}} \cdot \underbrace{\begin{bmatrix}-1\\1\\-1\\-1\end{bmatrix}}_{\mathbf{s}}\right]^{-1} \cdot \underbrace{\begin{bmatrix}-1 & 1 & -1 & -1\end{bmatrix}}_{\mathbf{s}^{\mathrm{T}}} = \begin{bmatrix}-1/4 & 1/4 & -1/4 & -1/4\end{bmatrix}$$

$$\Rightarrow k_1 = \underbrace{\begin{bmatrix}-1/4 & 1/4 & -1/4 & -1/4\end{bmatrix}}_{\mathbf{s}^{\mathrm{p}}} \cdot \underbrace{\begin{bmatrix}0\\1\\-2\\1\end{bmatrix}}_{\mathbf{s}} = \frac{1}{2}$$

c) $\hat{\mathbf{s}}_{n_1-1/2}(n_2,n_3-1) = \begin{bmatrix}{\scriptstyle 3} & 3 & {\scriptstyle 3} & 4 & {\scriptstyle 5} & 3 & {\scriptstyle 1} & 2 & {\scriptstyle 3} & 4 & {\scriptstyle 5}\end{bmatrix}^{\mathrm{T}}$

d) $\sum_{n_1=1}^{4} \left| s(n_1,n_2,n_3) - \hat{s}(n_1-\tfrac{1}{2},n_2,n_3-1)\right|^2 = |3-3|^2 + |4-4|^2 + |3-3|^2 + |2-2|^2 = 0$

absolute minimum reached – no improvement possible.

Additional Problem 4.18 (not in book)

A stereo camera system is used for depth analysis. The cameras are co-planar (parallel optical axes) (see Figure below), and the world coordinate system is defined with reference to the focal point, optical axis W_3 and image plane $W_3=-F$ of camera $\mathbf{C}_1$. The optical axes intersect the respective image planes exactly at the center, the sensor chips of each camera have an area of 10x7,5 mm², 400x300 pixel, and the focal length of both cameras is F=25 mm. Two points $\mathcal{P}_1$ and $\mathcal{P}_2$ are given on a planar surface, which is known to be vertically aligned in parallel with the image planes, i.e. the plane equation is

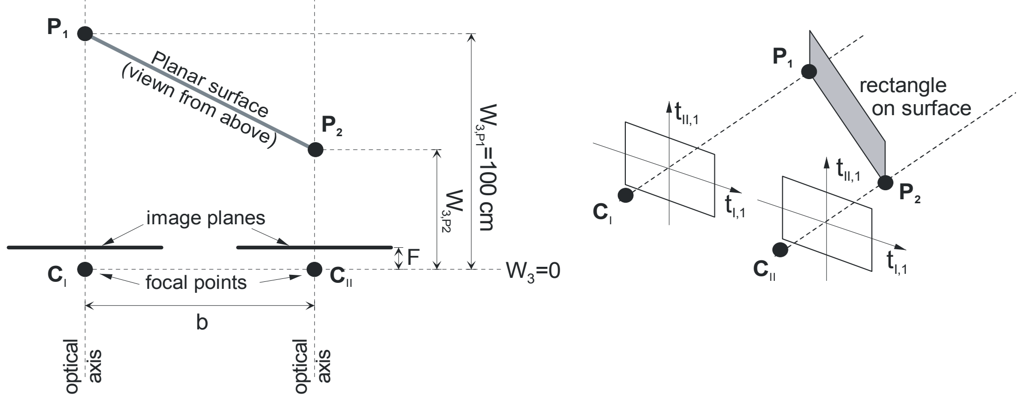
$$\frac{W_3}{W_{3,\mathcal{P}_1}} + \frac{W_1}{a} = 1 \quad \text{with} \quad a = b\frac{W_{3,\mathcal{P}_1}}{W_{3,\mathcal{P}_1} - W_{3,\mathcal{P}_2}} .$$

For calibration, the distance $W_{3,\mathcal{P}1}$=100 cm of $\mathcal{P}_1$ is known, it is also furthermore placed on the left camera's optical axis. For $\mathcal{P}_1$, a horizontal disparity $d_1(\mathcal{P}_1)$=100 pixel has been measured. Similarly, $\mathcal{P}_2$ is placed on the right camera's optical axis.

a) Determine the sampling distance T_1 and the stereoscopic parallax $t_{1,\mathrm{I},\mathcal{P}1} - t_{1,\mathrm{II},\mathcal{P}1}$.
b) Determine the baseline distance b.
c) For $\mathcal{P}_2$, the disparity measurement gives $d_2(\mathcal{P}_2)$=200 pixel. Determine $W_{3,\mathcal{P}2}$.
d) How would the disparities change with double or half focal lengths of the two cameras?

A rectangle of height H=5 cm is printed on the surface, with lower boundary at W_2=0. The lower boundary of the surface is at W_2=0.

e) Sketch images, how the rectangle would be projected into the left and right cameras' image planes. Is it possible to perform a mapping between the two images by the affine transform?



Arrangement of stereo cameras and points for which the distance shall be computed

Solution

a) $T_1 = \dfrac{10 \text{ mm}}{400} = 25\mu\text{m}; \quad t_{1,\mathrm{I},\mathscr{P}_1} - t_{1,\mathrm{II},\mathscr{P}_1} = d_1(\mathscr{P}_1)T_1 = 2.5 \text{ mm}$

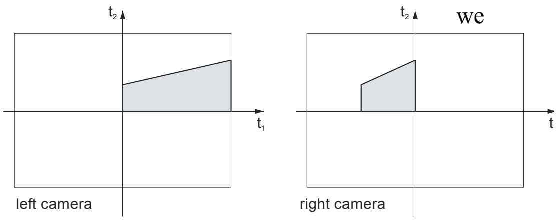
b) $$W_{3,\mathscr{P}_1} = F\frac{b}{t_{1,\mathrm{I},\mathscr{P}_1} - t_{1,\mathrm{II},\mathscr{P}_1}} \Rightarrow b = \frac{W_{3,\mathscr{P}_1}}{F}\left[t_{1,\mathrm{I},\mathscr{P}_1} - t_{1,\mathrm{II},\mathscr{P}_1}\right] = \frac{100 \text{ cm}}{25 \text{ mm}} \cdot 2.5 \text{ mm} = 10 \text{ cm}$$

c) $$t_{1,\mathrm{I},\mathscr{P}_2} - t_{1,\mathrm{II},\mathscr{P}_2} = d_1(\mathscr{P}_2)T_1 = 200 \cdot 25 \ \mu\text{m} = 5 \text{ mm}$$
$$\Rightarrow W_{3,\mathscr{P}_2} = F \cdot \frac{b}{t_{1,\mathrm{I},\mathscr{P}_2} - t_{1,\mathrm{II},\mathscr{P}_2}} = 25 \text{ mm} \cdot \frac{10 \text{ cm}}{5 \text{ mm}} = 50 \text{ cm}$$

d) $$W_{3,\mathscr{P}} = F\frac{b}{t_{1,\mathrm{I},\mathscr{P}} - t_{1,\mathrm{II},\mathscr{P}}} = F\frac{b}{d_1 T_1} \Rightarrow d_1 = F\frac{b}{W_{3,\mathscr{P}} T_1}$$

Disparity changes linearly with focal length, e.g. d**o**uble focal length gives double disparity.

e)



Affine mapping is possible, as the only modifications are translation (shift) and scaling of the horizontal axis. This is however a specific case due to the coplanar camera setup and the fact, that the rectanglular plane is also vertically parallet with the image planes. In general case, a perspective (or homography) mapping would be necessary.

Problem 5.1

a) Class centroid vectors: $\mathbf{m}^{(1)} = c \cdot \begin{bmatrix} 1.5 \\ 1.5 \end{bmatrix}; \ \mathbf{m}^{(2)} = \begin{bmatrix} 4.5 \\ 3 \end{bmatrix}; \ i) \ \mathbf{m} = \begin{bmatrix} 2.625 \\ 1.875 \end{bmatrix}; \ ii) \ \mathbf{m} = \begin{bmatrix} 3.75 \\ 3 \end{bmatrix}$

Covariance matrices:
$$\mathbf{C_{uu}}^{(1)} = \frac{1}{4} \cdot c^2 \cdot \left(\begin{bmatrix} 0.25 & 0.25 \\ 0.25 & 0.25 \end{bmatrix} + \begin{bmatrix} 0.25 & -0.25 \\ -0.25 & 0.25 \end{bmatrix} + \begin{bmatrix} 0.25 & -0.25 \\ -0.25 & 0.25 \end{bmatrix} + \begin{bmatrix} 0.25 & 0.25 \\ 0.25 & 0.25 \end{bmatrix} \right)$$
$$= c^2 \cdot \begin{bmatrix} 0.25 & 0 \\ 0 & 0.25 \end{bmatrix}$$

$$\mathbf{C_{uu}}^{(2)} = \frac{1}{4} \cdot \left(\begin{bmatrix} 0.25 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0.25 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2.25 & 1.5 \\ 1.5 & 1 \end{bmatrix} + \begin{bmatrix} 2.25 & 1.5 \\ 1.5 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1.25 & 0.75 \\ 0.75 & 0.5 \end{bmatrix}$$

$$i) \ \mathbf{C_{mm}} = \begin{bmatrix} 1.875 \\ 1.125 \end{bmatrix} \cdot \begin{bmatrix} 1.875 & 1.125 \end{bmatrix} = \begin{bmatrix} 3.516 & 2.109 \\ 2.109 & 1.266 \end{bmatrix}$$

$$ii) \ \mathbf{C_{mm}} = \begin{bmatrix} 0.75 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0.75 & 0 \end{bmatrix} = \begin{bmatrix} 0.5625 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{C_{uu}} = \frac{1}{2} \cdot \left(c^2 \cdot \begin{bmatrix} 0.25 & 0 \\ 0 & 0.25 \end{bmatrix} + \begin{bmatrix} 1.25 & 0.75 \\ 0.75 & 0.5 \end{bmatrix} \right)$$

$$\Rightarrow i) \ \mathbf{C_{uu}} = \begin{bmatrix} 0.657 & 0.375 \\ 0.375 & 0.281 \end{bmatrix} \quad ii) \ \mathbf{C_{uu}} = \begin{bmatrix} 1.125 & 0.375 \\ 0.375 & 0.75 \end{bmatrix}$$

$$\Rightarrow i) \ \left[\mathbf{C_{uu}}\right]^{-1} = \begin{bmatrix} 6.388 & -8.524 \\ -8.524 & 14.935 \end{bmatrix} \quad ii) \ \left[\mathbf{C_{uu}}\right]^{-1} = \begin{bmatrix} 1.067 & -0.533 \\ -0.533 & 1.600 \end{bmatrix}$$

Reliability criteria:

$$i) \ c_1 = \mathrm{tr}\left\{ \begin{bmatrix} 6.388 & -8.524 \\ -8.524 & 14.935 \end{bmatrix} \cdot \begin{bmatrix} 3.516 & 2.109 \\ 2.109 & 1.266 \end{bmatrix} \right\} = \mathrm{tr}\left\{ \begin{bmatrix} 4.483 & 2.681 \\ 1.527 & 0.931 \end{bmatrix} \right\} = 5.4$$

$$ii) \ c_1 = \mathrm{tr}\left\{ \begin{bmatrix} 1.067 & -0.533 \\ -0.533 & 0.281 \end{bmatrix} \cdot \begin{bmatrix} 0.5625 & 0 \\ 0 & 0 \end{bmatrix} \right\} = \mathrm{tr}\left\{ \begin{bmatrix} 0.6 & 0 \\ -0.3 & 0 \end{bmatrix} \right\} = 0.6$$

The second case indicates clearly worse classification reliability.

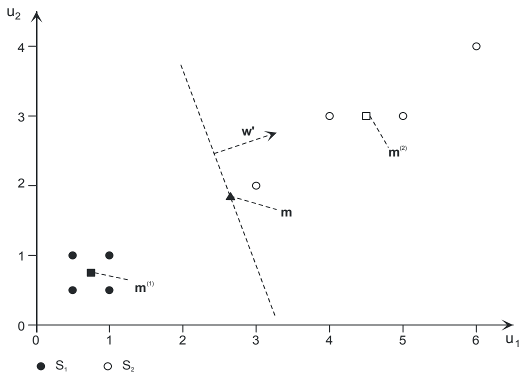
b) $\left[\mathbf{m}^{(1)}-\mathbf{m}^{(2)}\right]^{\mathrm{T}}\cdot\left[\mathbf{C}_{\mathbf{uu}}\right]^{-1}\cdot\left[\mathbf{u}_0-\dfrac{\mathbf{m}^{(1)}+\mathbf{m}^{(2)}}{2}\right]=0$ as Pr(S_1)=Pr(S_2)=0.5

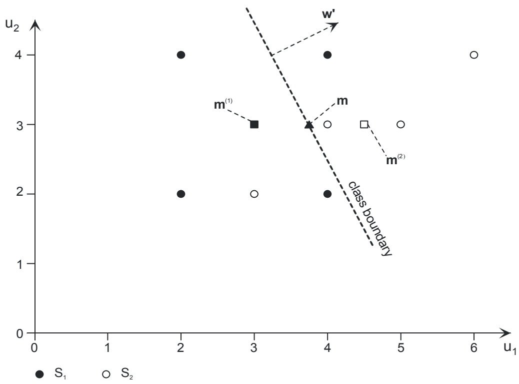
$$i)\ \begin{bmatrix}-3.75 & -2.25\end{bmatrix}\cdot\begin{bmatrix}6.388 & -8.524\\ -8.524 & 14.935\end{bmatrix}\cdot\left(\mathbf{u}_0-\begin{bmatrix}2.625\\ 1.875\end{bmatrix}\right)=\underbrace{\begin{bmatrix}-4.8 & -1.6\end{bmatrix}}_{\mathbf{w}^{\mathrm{T}}}\cdot\left(\mathbf{u}_0-\underbrace{\begin{bmatrix}2.625\\ 1.875\end{bmatrix}}_{\mathbf{m}}\right)=0$$

$$ii)\ \begin{bmatrix}-1.5 & 0\end{bmatrix}\cdot\begin{bmatrix}1.067 & -0.533\\ -0.533 & 1.600\end{bmatrix}\cdot\left(\mathbf{u}_0-\begin{bmatrix}3.75\\ 3\end{bmatrix}\right)=\underbrace{\begin{bmatrix}-1.6 & -0.8\end{bmatrix}}_{\mathbf{w}^{\mathrm{T}}}\cdot\left(\mathbf{u}_0-\underbrace{\begin{bmatrix}3.75\\ 3\end{bmatrix}}_{\mathbf{m}}\right)=0$$

The separation line crosses the point **m** and is perpendicular with **w**. In the following Figures, **w'** is a vector which points into the same direction as **w**.

c) c=0.5 : No wrong classification.

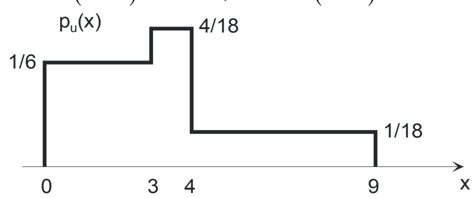




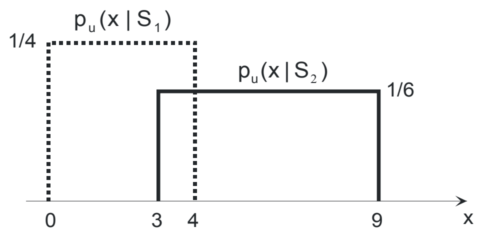
c=2 : $\mathbf{u}_4^{(1)}$ and $\mathbf{u}_4^{(2)}$ are wrongly classified.

Problem 5.2

a) $m^{(1)}=(4+0)/2=2$; $m^{(2)}=(9+3)/2=6$

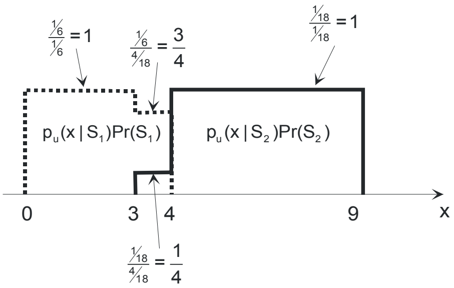


b) $\Pr(S_1)=\frac{1}{6}\cdot 4=\frac{2}{3}$; $\Pr(S_2)=\frac{1}{18}\cdot 6=\frac{1}{3}$; a priori probabilities: see Figure below.



c) Bayes rule :

$$\Pr(S_1\,|\,x)=\frac{p_u(x\,|\,S_1)\Pr(S_1)}{p_u(x)}\quad ;\quad \Pr(S_2\,|\,x)=\frac{p_u(x\,|\,S_2)\Pr(S_2)}{p_u(x)}$$



d) Optimum threshold θ=4, as $Pr(S_1|m) > Pr(S_2|m)$ for $m < 4$. No wrong classification in S_1. Probability of wrong classification in S_2: $Pr(S_1|S_2)=1/6$, overall probability of false classification is $Pr_{false}= Pr(S_1|S_2)\cdot Pr(S_2)=1/18$.

e) Condition : Same areas in the range of overlap:

$$\tfrac{1}{18}\cdot(\theta_= - 3) = \tfrac{1}{6}\cdot(4-\theta_=) \Rightarrow \theta_= = 3.75$$

$\Rightarrow Pr(S_2|S_1)=1/16$, $Pr(S_1|S_2)=1/8$, $Pr_{false}= Pr(S_2|S_1)\cdot Pr(S_1)+Pr(S_1|S_2)\cdot Pr(S_2)=2/48+1/24=1/12$

Problem 5.3

a) $\mathbf{m}^{(1)} = \begin{bmatrix} 2.5 \\ 2.5 \end{bmatrix}$; $\mathbf{m}^{(2)} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$; $\mathbf{m}^{(3)} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$

Covariances are zero due to statistical independency of the features (joint probability is product of the first-order probabilities). Variances within all classes are identical for features u_1 and u_2:

$$\sigma_{u_1}^{2\,(1)} = \sigma_{u_2}^{2\,(1)} = \frac{3^2}{12} \;;\; \sigma_{u_1}^{2\,(2)} = \sigma_{u_2}^{2\,(2)} = \frac{4^2}{12} \;;\; \sigma_{u_1}^{2\,(3)} = \sigma_{u_2}^{2\,(3)} = \frac{2^2}{12}$$

$$\mathbf{C_{uu}}^{(1)} = \begin{bmatrix} \frac{3}{4} & 0 \\ 0 & \frac{3}{4} \end{bmatrix} \;;\; \mathbf{C_{uu}}^{(2)} = \begin{bmatrix} \frac{4}{3} & 0 \\ 0 & \frac{4}{3} \end{bmatrix} \;;\; \mathbf{C_{uu}}^{(3)} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

b)

$$p_u(\mathbf{x}\,|\,S_1) = \frac{1}{9} \;\Rightarrow\; p_u(\mathbf{x}\,|\,S_1)\Pr(S_1) = \frac{1}{36}$$
$$p_u(\mathbf{x}\,|\,S_2) = \frac{1}{16} \;\Rightarrow\; p_u(\mathbf{x}\,|\,S_2)\Pr(S_2) = \frac{3}{80}$$
$$p_u(\mathbf{x}\,|\,S_3) = \frac{1}{4} \;\Rightarrow\; p_u(\mathbf{x}\,|\,S_3)\Pr(S_3) = \frac{3}{80}$$

c)

$$A: \Pr(S_1\,|\,\mathbf{x}) = \frac{p_\mathbf{u}(\mathbf{x}\,|\,S_1)\Pr(S_1)}{p_\mathbf{u}(\mathbf{x}\,|\,S_1 \cup S_2)} = \frac{\frac{1}{36}}{\frac{1}{36}+\frac{3}{80}} = \frac{1}{1+\frac{27}{20}} = \frac{20}{47} \;;\; \Pr(S_2\,|\,\mathbf{x}) = \frac{p_\mathbf{u}(\mathbf{x}\,|\,S_2)\Pr(S_2)}{p_\mathbf{u}(\mathbf{x}\,|\,S_1 \cup S_2)} = \frac{1}{1+\frac{20}{27}} = \frac{27}{47}$$

$$B: \Pr(S_2\,|\,\mathbf{x}) = \frac{p_\mathbf{u}(\mathbf{x}\,|\,S_2)\Pr(S_2)}{p_\mathbf{u}(\mathbf{x}\,|\,S_2 \cup S_3)} = \frac{\frac{3}{80}}{\frac{3}{80}+\frac{3}{80}} = \frac{1}{2} \;;\; \Pr(S_3\,|\,\mathbf{x}) = \frac{p_\mathbf{u}(\mathbf{x}\,|\,S_3)\Pr(S_3)}{p_\mathbf{u}(\mathbf{x}\,|\,S_2 \cup S_3)} = \frac{\frac{3}{80}}{\frac{3}{80}+\frac{3}{80}} = \frac{1}{2}$$

d) MAP classification: Within the area "A" $Pr(S_2|\mathbf{m})>Pr(S_1|\mathbf{m})$, hence all S_1 are wrongly classified here. Area "A" is 1/9 of the entire area of $S_1 \Rightarrow Pr(S_1|S_2)=0$, $Pr(S_2|S_1)=1/9$. Within the area "B" $Pr(S_2|\mathbf{m})=Pr(S_3|\mathbf{m})$: Half of all S_2 and S_3 within this area are wrongly classified $\Rightarrow Pr(S_2|S_3)=1/2\cdot 1/4=1/8$, $Pr(S_3|S_2)=1/2\cdot 1/16=1/32$
Total classification error:
$Prob_{err}= Pr(S_2|S_1)Pr(S_1) + Pr(S_3|S_2)Pr(S_2) + Pr(S_2|S_3)Pr(S_3)=1/36+3/160+3/160=47/720$

Additional Problem 5.4 (not in book)

The features of two classes are statistically described by vector Gaussian distributions with the following parameters:

$$\mathbf{C_{uu}}^{(1)} = \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix} \;;\; \mathbf{m}^{(1)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \;;\; \mathbf{C_{uu}}^{(2)} = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix} \;;\; \mathbf{m}^{(2)} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

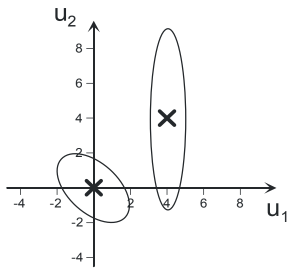
a) Sketch qualitatively the orientations of ellipses, which characterize the feature clusters of the two classes.

The feature vector $\mathbf{u} = [1 \quad 4]^T$ shall be classified.

b) Determine the class allocation according to Euclidean distance criterion.
c) Determine the class allocation according to Mahalanobis distance criterion.
d) Describe a covariance matrix $\mathbf{C_{uu}}^{(2)}$, such that for arbitrary feature vectors the classification results for both distance criteria of b) and c). Also for this case, determine and sketch the positions of the separation line in the feature space.

Solution

a) Class 1: Negative correlation, equal variances of features, therefore orientation of ellipse by -45^0. Eigenvalues of covariance matrix λ_1=4(1+1/2)=6 and λ_2=4(1−1/2)=2.
Class 2: Uncorrelated, variance of features σ_1^2=1, σ_2^2=10. Ellipse axis lengths are proportional with square roots of eigenvalues, respectively standard devations.



b) $\left[\mathbf{u}-\mathbf{m}^{(1)}\right]^{\mathrm{T}}\left[\mathbf{u}-\mathbf{m}^{(1)}\right]=1+16=17 \quad ; \quad \left[\mathbf{u}-\mathbf{m}^{(2)}\right]^{\mathrm{T}}\left[\mathbf{u}-\mathbf{m}^{(2)}\right]=9+0=9$

Class 2 is the better choice.

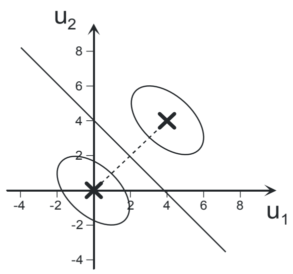
$$\left[\mathbf{C_{uu}}^{(1)}\right]^{-1}=\frac{1}{12}\begin{bmatrix}4 & 2\\ 2 & 4\end{bmatrix}=\begin{bmatrix}1/3 & 1/6\\ 1/6 & 1/3\end{bmatrix} \quad ; \quad \left[\mathbf{C_{uu}}^{(2)}\right]^{-1}=\begin{bmatrix}1 & 0\\ 0 & 1/10\end{bmatrix}$$

c) $\left[\mathbf{u}-\mathbf{m}^{(1)}\right]^{\mathrm{T}}\left[\mathbf{C_{uu}}^{(1)}\right]^{-1}\left[\mathbf{u}-\mathbf{m}^{(1)}\right]=1/3+2/3+2/3+16/3=7$

$\left[\mathbf{u}-\mathbf{m}^{(2)}\right]^{\mathrm{T}}\left[\mathbf{C_{uu}}^{(2)}\right]^{-1}\left[\mathbf{u}-\mathbf{m}^{(2)}\right]=9+0=9$

Now, class 1 is the better choice.

d) For the case of Euclidean distance, the separation line is a Voronoi line. To achive the same result with Mahalanobis distance, the covariance matrices must be identical, and one of the ellipse axes must be perpendicular with the Delaunay line that connects the class centroids. As the latter condition is fulfilled for the principal axis in the case of class 1, the condition $\mathbf{C_{uu}}^{(2)}=\mathbf{C_{uu}}^{(1)}$ is sufficient.



Problem 6.1

Note: Instead of ‘L’ and ‘H’ as used for expressing low and high amplitude classes in Sec. 6.1.1 of the book, S_l with l=1,2 is used in this problem. Herein, classes l=1 and l=2 represent the low and high amplitudes, respectively.

a) Class assignments S_l for cases

Θ=1.5: Θ=3: Θ=4.5 :

$$\mathbf{S}_l=\begin{bmatrix}1&2&2&2&2\\1&1&2&2&2\\1&2&2&2&2\\1&1&2&2&2\\1&1&1&2&2\end{bmatrix} \qquad \mathbf{S}_l=\begin{bmatrix}1&1&2&2&2\\1&1&1&2&2\\1&1&2&2&2\\1&1&1&2&2\\1&1&1&1&2\end{bmatrix} \qquad \mathbf{S}_l=\begin{bmatrix}1&1&1&2&2\\1&1&1&1&2\\1&1&1&2&2\\1&1&1&1&2\\1&1&1&1&1\end{bmatrix}$$

b) Overall mean: $m_s=\frac{1}{25}\cdot\left[9\cdot1+5\cdot2+5\cdot4+6\cdot5\right]=2.76$

<u>Θ=1.5 :</u>

$$m_{S_1} = 1\,;\sigma_{S_1}^2 = 0\,;\; m_{S_2} = \frac{1}{16}\cdot\left[5\cdot 2+5\cdot 4+6\cdot 5\right] = 3.75\,;\sigma_{S_2}^2 = \frac{1}{16}\cdot\left[5\cdot 2^2+5\cdot 4^2+6\cdot 5^2\right]-3.75^2 = 1.5625$$

$$\Pr(S_1) = \frac{9}{25} = 0.36\,;\; \Pr(S_2) = \frac{16}{25} = 0.64$$

$$\Rightarrow \sigma_{\mathrm{m}}{}^2 = 0.36\cdot(1-2.76)^2 + 0.64\cdot(3.75-2.76)^2 = 1.7424\,;\; \bar{\sigma}^2 = 0.36\cdot 0 + 0.64\cdot 1.5625 = 1$$

$$\Rightarrow \frac{\sigma_{\mathrm{m}}{}^2}{\bar{\sigma}^2} = 1.7424$$

<u>Θ=3 :</u>

$$m_{S_1} = \frac{1}{14}\cdot\left[9\cdot 1+5\cdot 2\right] = 1.357\,;\sigma_{S_1}^2 = \frac{1}{14}\cdot\left[9\cdot 1^2+5\cdot 2^2\right]-1.357^2 = 0.23$$

$$m_{S_2} = \frac{1}{11}\cdot\left[5\cdot 4+6\cdot 5\right] = 4.545\,;\sigma_{S_2}^2 = \frac{1}{11}\cdot\left[5\cdot 4^2+6\cdot 5^2\right]-4.545^2 = 0.248$$

$$\Pr(S_1) = \frac{14}{25} = 0.56\,;\; \Pr(S_2) = \frac{11}{25} = 0.44$$

$$\Rightarrow \sigma_{\mathrm{m}}{}^2 = 0.56\cdot(1.375-2.76)^2 + 0.44\cdot(4.545-2.76)^2 = 2.476\,;$$

$$\bar{\sigma}^2 = 0.56\cdot 0.23 + 0.44\cdot 0.248 = 0.238 \Rightarrow \frac{\sigma_{\mathrm{m}}{}^2}{\bar{\sigma}^2} = 10.40$$

<u>Θ=4.5 :</u>

$$m_{S_1} = \frac{1}{19}\cdot\left[9\cdot 1+5\cdot 2+5\cdot 4\right] = 2.053\,;\sigma_{S_1}^2 = \frac{1}{19}\cdot\left[9\cdot 1^2+5\cdot 2^2+5\cdot 4^2\right]-2.053^2 = 1.522\,;\; m_{S_2} = 5\,;\sigma_{S_2}^2 = 0$$

$$\Pr(S_1) = \frac{19}{25} = 0.76\,;\; \Pr(S_2) = \frac{9}{25} = 0.24$$

$$\Rightarrow \sigma_{\mathrm{m}}{}^2 = 0.76\cdot(2.053-2.76)^2 + 0.24\cdot(5-2.76)^2 = 1.584\,;$$

$$\bar{\sigma}^2 = 0.76\cdot 1.522 + 0.24\cdot 0 = 1.157 \Rightarrow \frac{\sigma_{\mathrm{m}}{}^2}{\bar{\sigma}^2} = 1.369$$

Optimum threshold segmentation is achieved for Θ=3.

c)

$$\Pr(S_2 \mid S_1) = \frac{1}{\sqrt{2}\sigma_{S_1}}\int_{\Theta}^{\infty} \mathrm{e}^{-\frac{\sqrt{2}\left|x-m_{S_1}\right|}{\sigma_{S_1}}}\,\mathrm{d}x = \frac{1}{\sqrt{2}\sigma_{S_1}}\int_{\Theta-m_{S_1}}^{\infty} \mathrm{e}^{-\frac{\sqrt{2}x}{\sigma_{S_1}}}\,\mathrm{d}x$$

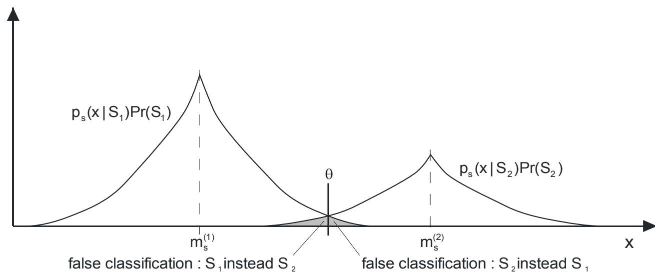
$$= \frac{1}{\sqrt{2}\sigma_{S_1}}\left(-\frac{\sigma_{S_1}}{\sqrt{2}}\right)\left|\,\mathrm{e}^{-\frac{\sqrt{2}x}{\sigma_{S_1}}}\right|_{\Theta-m_{S_1}}^{\infty} = \frac{1}{2}\mathrm{e}^{-\frac{\sqrt{2}\left(\Theta-m_{S_1}\right)}{\sigma_{S_1}}}$$

$$\Pr(S_1 \mid S_2) = \frac{1}{\sqrt{2}\sigma_{S_2}}\int_{-\infty}^{\Theta} \mathrm{e}^{-\frac{\sqrt{2}\left|x-m_{S_2}\right|}{\sigma_{S_2}}}\,\mathrm{d}x = \frac{1}{\sqrt{2}\sigma_{S_2}}\int_{m_{S_2}-\Theta}^{\infty} \mathrm{e}^{-\frac{\sqrt{2}x}{\sigma_{S_2}}}\,\mathrm{d}x$$

$$= \frac{1}{\sqrt{2}\sigma_{S_2}}\left(-\frac{\sigma_{S_2}}{\sqrt{2}}\right)\left|\,\mathrm{e}^{-\frac{\sqrt{2}x}{\sigma_{S_2}}}\right|_{m_{S_2}-\Theta}^{\infty} = \frac{1}{2}\mathrm{e}^{-\frac{\sqrt{2}\left(m_{S_2}-\Theta\right)}{\sigma_{S_2}}}$$

$$\mathrm{Prob}_{\mathrm{err}} = \Pr(S_2 \mid S_1)\Pr(S_1) + \Pr(S_1 \mid S_2)\Pr(S_2)\,;$$

Θ=3 ⇒ $\mathrm{Prob}_{\mathrm{err}}$=0.56·0.0039+0.44·0.0062=0.0049 ≈0.5 %



Problem 6.2

Note: Segment class indexing S_l with l=1,2,3 is used in this problem. Herein, the amplitude increases with higher class index l.

a)
$$\mathbf{I}_A = \begin{bmatrix} 1 & 2 & 2 & 2 & 3 \\ 1 & 2 & 2 & 2 & 3 \\ 1 & 2 & 2 & 2 & 3 \end{bmatrix} \quad ; \quad \mathbf{I}_B = \begin{bmatrix} 1 & 1 & 2 & 3 & 3 \\ 1 & 1 & 2 & 3 & 3 \\ 1 & 1 & 2 & 3 & 3 \end{bmatrix}$$

b) Θ_A: Pr(S_1)=1/5; Pr(S_2)=3/5; Pr(S_3)=1/5; m_{S_1}=1; m_{S_2}=3; m_{S_3}=5; $\sigma_{S_1}{}^2$=0; $\sigma_{S_2}{}^2$=2/3; $\sigma_{S_3}{}^2$=0.

Θ_B: Pr(S_1)=2/5; Pr(S_2)=1/5; Pr(S_3)=2/5; m_{S_1}=1,5; m_{S_2}=2; m_{S_3}=4,5; $\sigma_{S_1}{}^2$=1/4; $\sigma_{S_2}{}^2$=0; $\sigma_{S_3}{}^2$=1/4.

c)
$$\overline{\sigma^2}(\boldsymbol{\theta}_A) = \frac{1}{5}\cdot 0 + \frac{3}{5}\cdot\frac{2}{3} + \frac{1}{5}\cdot 0 = \frac{2}{5};\ \sigma_m{}^2(\boldsymbol{\theta}_B) = \frac{1}{5}\cdot(1-3)^2 + \frac{3}{5}\cdot(3-3)^2 + \frac{1}{5}\cdot(5-3)^2 = \frac{8}{5}$$
$$\overline{\sigma^2}(\boldsymbol{\theta}_B) = \frac{2}{5}\cdot\frac{1}{4} + \frac{1}{5}\cdot 0 + \frac{2}{5}\cdot\frac{1}{4} = \frac{1}{5};\ \sigma_m{}^2(\boldsymbol{\theta}_B) = \frac{2}{5}\cdot(1{,}5-3)^2 + \frac{1}{5}\cdot(3-3)^2 + \frac{2}{5}\cdot(4{,}5-3)^2 = \frac{9}{5}$$
$\sigma_m{}^2(\boldsymbol{\theta}_A)/\overline{\sigma^2}(\boldsymbol{\theta}_A) = 4$; $\sigma_m{}^2(\boldsymbol{\theta}_A)/\overline{\sigma^2}(\boldsymbol{\theta}_A) = 9$, Θ_B is the better choice.

d) With Θ_C=[1 5]^T, Pr(S_1)=0; P(S_2)=1; Pr(S_3)=0; m_{S_2}=3; $\sigma_{S_2}{}^2$=7/5 $\Rightarrow \dfrac{\sigma_m{}^2(\boldsymbol{\theta}_C)}{\overline{\sigma^2}(\boldsymbol{\theta}_C)} = \dfrac{0}{7/5} = 0$, which is the lowest possible value.