Jens-Rainer Ohm

Multimedia Signal Coding and Transmission

Solutions to End-of-chapter Problems

Equations for **inversion** of 2x2 and 3x3 matrices:

$$\mathbf{A}^{-1}=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1}=\frac{1}{a_{11}a_{22}-a_{12}a_{21}}\cdot\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

$$\mathbf{A}^{-1}=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^{-1}=\frac{1}{|\mathbf{A}|}\cdot\begin{bmatrix} a_{22}a_{33}-a_{23}a_{32} & a_{13}a_{32}-a_{12}a_{33} & a_{12}a_{23}-a_{13}a_{22} \\ a_{31}a_{23}-a_{21}a_{33} & a_{11}a_{33}-a_{13}a_{31} & a_{13}a_{21}-a_{11}a_{23} \\ a_{21}a_{32}-a_{31}a_{22} & a_{12}a_{31}-a_{11}a_{32} & a_{11}a_{22}-a_{12}a_{21} \end{bmatrix}$$

$$|\mathbf{A}|=a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}-a_{13}a_{22}a_{31}$$

Problem 1.1

a) per image frame: 1920 x 1080 + 2 · (960 x 1080) Byte = 4.147,200 Byte
per second: 4.147,200 Byte · 60 = 248.832 Mbyte
per minute: 248.832 Mbyte · 60 = 14.92992 Gbyte
whole movie: 14.92992 Gbyte · 150 ≈ 2.2395 Tbyte

b) 2239.5 x (1:200) = 11.1975 GByte

c) 248.832 Mbyte/s = 248.832 · 8 Mbit/s = 1.990656 Gbit/s
1.990656 Gbit/s · (1:200) = 9.95328 Mbit/s using HEVC
Increase by 10%: 9.95328 Mbit/s · 1.1 = 10.9486 Mbit/s

Note: The calculations above define 1 Mbyte as 1.000,000 Byte. In calculating hard disc storage, typically the definition 1 MByte as $1{,}024^2$=1.048,576 Byte is used.

Problem 1.2

a) Rate for video signal: 720 kbit/s – 20 kbit/s = 700 kbit/s
buffer memory size: 700 kbit/s · 0.1 s = 70 kbit

b) Mean number of bits per frame is 700 kbit/s : 30 frames/s = 23.333 kbit/frame (target data rate). Decrease of Q factor by k effects increase of actual data rate by $(1.1)^k$ – if the Q factor is increased, k is less than zero, such that the data rate becomes lower.
Effect of the rate control: updated data rate = actual data rate · $(1.1)^k$ ≤target
19.28 kbit · $(1.1)^k \le$ 23.33 kbit ⇒ $(1.1)^k \le$ 23.33/19.28 kbit ⇒ $k \cdot \log(1.1) \le \log(23.33/19.28)$
$k \le \log(23.33/19.28) / \log(1.1) \approx 2.154$

The best choice is decreasing the Q factor by two steps, which provides better quality. This may bring up the rate to approximately 23.3288 kbit/frame or 699.86 kbit/s.

Note: Practically, a more conservative choice, e.g. decreasing by only one step may be more appropriate. Firstly, the rule of thumb given here may not appy to all sequences; second, strong fluctuations of quality may be undesirable; third, if there are different picture types in encoding (such as intra, unidirectional and bidirectional prediction), their bit consumption may also be quite different.

Problem 2.1

a) With the frequency domain sampling basis $\mathbf{F}=\frac{1}{T}\begin{bmatrix} \frac{\sqrt{3}}{2} & 0 \\ -\frac{1}{2} & 1 \end{bmatrix}$

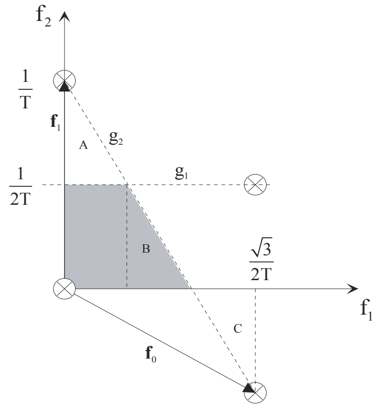
Interpretation from the following Figure: Boundary of base band (gray) in the first quadrant is limited by lines g_1 and g_2. The centers of basebands and of next periodic spectrum copies ⊗ establish corners of triangles having side lengths $|\mathbf{f}_0|=|\mathbf{f}_1|=|1/T|$. Triangles A,B and C are congruent.

The intercepts of the lines are as given in the following table:

	f_1 axis	f_2 axis
g_1	parallel	$1/(2T)$
g_2	$\sqrt{3}/(3T)$	$1/T$

The line equations are

g1 : $f_2 = \frac{1}{2T}$



g2 : $\frac{f_1 T}{\sqrt{3}/3} + f_2 T = 1 \quad \Leftrightarrow \quad f_1 + \frac{f_2}{\sqrt{3}} = \frac{1}{\sqrt{3}T}$

All signal components above any of the two lines must be zero to fulfill the sampling conditions. In the first quadrant, this can be formulated as

$S(f_1, f_2) = 0 \;\text{ for }\; f_2 \geq \frac{1}{2T} \;\text{ and }\; f_1 + \frac{f_2}{\sqrt{3}} \geq \frac{1}{\sqrt{3}T}$.

Generalization into all four quadrants gives (2.72).

b) From Figure above: Gray area in first quadrant (rectangle + triangle B having half size of rectangle)

$$A_{\text{hex}} = \frac{1}{2T} \cdot \frac{\sqrt{3}}{6T} \cdot 1.5 = \frac{1}{4T^2} \cdot \frac{\sqrt{3}}{2}$$

In comparison, using rectangular sampling with $\omega_R = \omega_S$ in the first quadrant (from Fig. 2.13c)

$$A_{\text{rect}} = \left(\frac{1}{2T}\right)^2 \Rightarrow \frac{A_{\text{hex}}}{A_{\text{rect}}} = \frac{\sqrt{3}}{2} \approx 0.866$$

c) From Fig. 2.13a resp. length of the basis vector $\mathbf{t}_0$ from (2.63) : $T_{1,\text{hex}} = \frac{2}{\sqrt{3}} T_2$.

For rectangular sampling $T_{1,\text{rect}} = T_2 \quad \Rightarrow \frac{T_{1,\text{hex}}}{T_{1,\text{rect}}} = \frac{2}{\sqrt{3}}$.

d) $\frac{|\mathbf{T}_{\text{hex}}|}{T^2} = \frac{2}{\sqrt{3}} \cdot 1 - \frac{1}{\sqrt{3}} \cdot 0 = \frac{2}{\sqrt{3}} = \frac{T_{1,\text{hex}}}{T_{1,\text{rect}}} = \frac{A_{\text{rect}}}{A_{\text{hex}}}$

Results of c) and d) determine the factor, by which the number of samples is lower in case of hexagonal sampling, as compared to rectangular sampling using identical line spacing. The result from b) is the reciprocal value, as the size of the baseband spectrum behaves reciprocally with the 'sampling area'.

e) Area of the CCD chip 10x7.5 mm^2, $N_1 N_2 = 4 \cdot 10^6$ Pixel.

Rectangular: $T_1 = T_2 \Rightarrow N_{1,\text{rect}} = \frac{10}{7.5} N_{2,\text{rect}} \Rightarrow \frac{4}{3}\left(N_{2,\text{rect}}\right)^2 = 4 \cdot 10^6$

$\Rightarrow N_{2,\text{rect}} = \sqrt{3} \cdot 10^3 \approx 1{,}732$ rows

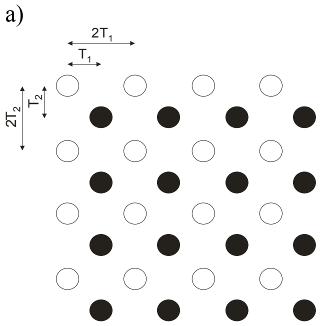
Hexagonal: $T_1 = \frac{2}{\sqrt{3}} T_2 \Rightarrow N_{1,\text{hex}} = \frac{10}{7.5} \cdot \frac{\sqrt{3}}{2} \cdot N_{2,\text{hex}} \Rightarrow \frac{2}{\sqrt{3}}\left(N_{2,\text{hex}}\right)^2 = 4 \cdot 10^6$

$\Rightarrow N_{2,\text{hex}} = \sqrt{2 \cdot \sqrt{3}} \cdot 10^3 \approx 1{,}861$ rows

Sampling distances:
rectangular 7.5 mm / 1,732 = 4.33 μm ; hexagonal 7.5 mm / 1,861 = 4.03 μm.

Problem 2.2

a)



b)

$$S_\delta(f_1,f_2)=S(f_1,f_2)**\left[\frac{1}{4T_1T_2}\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\delta\left(f_1-\frac{k_1}{2T_1},f_2-\frac{k_2}{2T_2}\right)\left(1+\mathrm{e}^{-\mathrm{j}2\pi(f_1T_1+f_2T_2)}\right)\right]$$

$$=S(f_1,f_2)**\left[\frac{1}{4T_1T_2}\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\delta\left(f_1-\frac{k_1}{2T_1},f_2-\frac{k_2}{2T_2}\right)\underbrace{\left(1+\mathrm{e}^{-\mathrm{j}\pi(k_1+k_2)}\right)}_{\substack{=2,k_1+k_2=2k_2' \text{ even}\\ =0,k_1+k_2 \text{ odd}}}\right]$$

$$=S(f_1,f_2)**\left[\frac{1}{2T_1T_2}\sum_{k_1=-\infty}^{\infty}\sum_{k_2'=-\infty}^{\infty}\delta\left(f_1-\frac{k_1}{2T_1},f_2-\frac{k_2'}{T_2}+\frac{k_1}{2T_2}\right)\right]$$

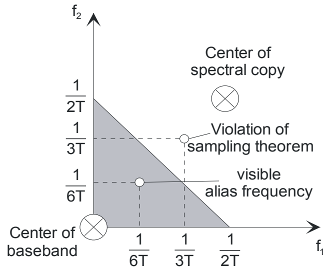
$$=|\mathbf{F}|\sum_{\mathbf{k}}S(\mathbf{f}-\mathbf{kF})\ \text{ with }\ T=T_1=T_2\ \text{ and }\ \mathbf{F}=\frac{1}{T}\begin{bmatrix}1/2 & 0\\ -1/2 & 1\end{bmatrix}$$

Note: With different scaling $T=T_2=T_1\sqrt{3}/2$ this would give the scheme of hexagonal sampling, Fig. 2.11c and (2.63) top.

Problem 2.3

a) Boundaries of the baseband: Boundary $|f_1|+|f_2|=\frac{1}{2T}\Rightarrow\frac{1}{3T}+|F_2|<\frac{1}{2T}\Rightarrow|F_2|<\frac{1}{6T}$.

b) According to the result from a), the sampling condition is violated. Alias in the first quadrant results from the spectral copy at F_1=1/(2T), F_2=1/(2T). The resulting alias frequency is $F_1'=1/(2T)-1/(3T)=1/(6T)$, $F_2'=1/(2T)-1/(3T)=1/(6T)$.



Problem 2.4

$$p_s(x)=a\,\mathrm{e}^{-|bx|^\gamma}\quad;\quad a=\frac{b\gamma}{2\Gamma\left(\frac{1}{\gamma}\right)}\quad;\quad b=\frac{1}{\sigma_s}\sqrt{\frac{\Gamma\left(\frac{3}{\gamma}\right)}{\Gamma\left(\frac{1}{\gamma}\right)}}\quad;\quad\Gamma(x)=\int_0^\infty\mathrm{e}^{-t}\,y^{x-1}\,\mathrm{d}\,y$$

a) $$b=\frac{1}{\sigma_s}\sqrt{\frac{\left(\frac{\sqrt{\pi}}{2}\right)}{\sqrt{\pi}}}=\frac{1}{\sigma_s}\sqrt{\frac{1}{2}}=\frac{1}{\sqrt{2\sigma_s^2}}\ ;\ a=\frac{\frac{1}{\sqrt{2\sigma_s^2}}2}{2\sqrt{\pi}}=\frac{1}{\sqrt{2\pi\sigma_s^2}}$$

$$\Rightarrow p_s(x)=\frac{1}{\sqrt{2\pi\sigma_s^2}}\mathrm{e}^{-\frac{x^2}{2\sigma_s^2}}$$

b) $b=\frac{1}{\sigma_s}\sqrt{\frac{2}{1}}=\frac{\sqrt{2}}{\sigma_s}$; $a=\frac{\left(\frac{\sqrt{2}}{\sigma_s}\right)}{2}=\frac{1}{\sqrt{2}\sigma_x}\Rightarrow p_s(x)=\frac{1}{\sqrt{2}\sigma_s}\mathrm{e}^{-\frac{\sqrt{2}\cdot|x|}{\sigma_s}}$

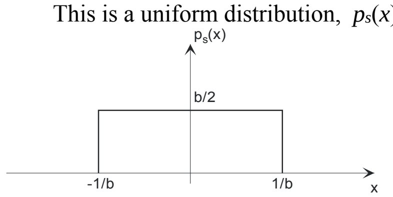
c) $\lim_{x\to\infty}\Gamma\left(\frac{3}{x}\right)=\Gamma(0)$; $\lim_{x\to\infty}\Gamma\left(\frac{1}{x}\right)=\Gamma(0)$; but $\Gamma(0)=\lim_{x\to 0}\frac{\Gamma(1)}{x}=\lim_{x\to 0}\frac{1}{x}=\infty$

$$b=\frac{1}{\sigma_s}\sqrt{\frac{\Gamma(3/\infty)}{\Gamma(1/\infty)}}=?$$

Assumption : b is a positive and real-valued constant. Different cases are:

$|x|<\frac{1}{b}\Rightarrow|bx|<1\Rightarrow\lim_{\gamma\to\infty}|bx|^\gamma=0\Rightarrow\lim_{\gamma\to\infty}e^{-|bx|^\gamma}=1$ und $|x|>\frac{1}{b}\Rightarrow|bx|>1\Rightarrow\lim_{\gamma\to\infty}|bx|^\gamma=\infty\Rightarrow\lim_{\gamma\to\infty}\mathrm{e}^{-|bx|^\gamma}=0$

This is a uniform distribution, $p_s(x)=a=b/2$ for $-1/b<x<1/b$.



The constants a and b are: $\sigma_s^2=\frac{b}{2}\int_{-1/b}^{1/b}x^2dx=\frac{1}{3b^2}\Rightarrow b=\frac{1}{\sqrt{3}\sigma_s}$; $a=\frac{b}{2}=\frac{1}{\sqrt{12}\sigma_s}$

Problem 2.5

a) $\mu_{ss}(k)=\sigma_s^2\rho^{|k|}\Rightarrow\mu_{ss}(0)=\sigma_s^2$; $\mu_{ss}(1)=\sigma_s^2\rho$; $\mu_{ss}(2)=\sigma_s^2\rho^2$ $\quad\mathbf{C_{ss}}=\sigma_s^2\begin{bmatrix}1&\rho&\rho^2\\\rho&1&\rho\\\rho^2&\rho&1\end{bmatrix}$

b) $\mathbf{C_{ss}}=\sigma_s^2\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\Rightarrow\mathbf{C_{ss}}^{-1}=\frac{1}{\sigma_s^2}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ $\quad$ Det$|\mathbf{C_{ss}}|=[\sigma_s^2]^3$

$$p_{\mathbf{s},3}(\mathbf{x})=\frac{1}{\sqrt{(2\pi)^3\left(\sigma_s^2\right)^3}}\mathrm{e}^{-\frac{1}{2\sigma_s^2}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}^{\mathrm{T}}\cdot\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\cdot\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}}=\frac{1}{\sqrt{(2\pi)^3\left(\sigma_s^2\right)^3}}\mathrm{e}^{-\frac{1}{2\sigma_s^2}\left[x_1^2+x_2^2+x_3^2\right]}$$

$$=\frac{1}{\left[\sqrt{2\pi\sigma_s^2}\right]^3}\mathrm{e}^{-\frac{x_1^2}{2\sigma_s^2}}\mathrm{e}^{-\frac{x_2^2}{2\sigma_s^2}}\mathrm{e}^{-\frac{x_3^2}{2\sigma_s^2}}=p_s(x_1)p_s(x_2)p_s(x_3)$$

Problem 2.6

a) For statistical independency, $\Pr(j_1,j_2)=\Pr(j_1)\Pr(j_2)$, $\Pr(j_1|j_2)=\Pr(j_1)$, $\Pr(j_2|j_1)=\Pr(j_2)$. Hence,

$$H\left(S_2\,|\,S_1\right)=-\sum_{j_1}\sum_{j_2}\Pr(j_1)\Pr(j_2)\log_2\Pr(j_2)$$
$$=-\underbrace{\sum_{j_1}\Pr(j_1)}_{=1}\sum_{j_2}\Pr(j_2)\log_2\Pr(j_2)=H(S_2)$$

similarly for $H(j_1|j_2)$. This gives $I(S_1;S_2)=H(S_2)-H(S_2\,|\,S_1)=H(S_1)-H(S_1\,|\,S_2)=0$,

which can also be shown by direct computation:

$$I(S_1;S_2)=\sum_{j_1}\sum_{j_2}\Pr(j_1)\Pr(j_2)\log_2\frac{\Pr(j_1)\Pr(j_2)}{\Pr(j_1)\Pr(j_2)}=0$$

b) For this case, $\Pr(j_1,j_2)=\Pr(j_1)\,\delta(j_1-j_2)=\Pr(j_2)\,\delta(j_1-j_2)$, $\Pr(j_1|j_2)=\Pr(j_2|j_1)=\delta(j_1-j_2)$ (discrete delta impulse: $=0$ for $j_1\neq j_2$, $=1$ else). Hence, all entries $j_1\neq j_2$ can be disregarded:

$$H(S_1\,|\,S_2)=\sum_{j_1(j_1=j_2)}\Pr(j_1)\log_2(1)=0\Rightarrow I(S_1;S_2)=H(S_1)-H(S_1\,|\,S_2)=H(S_1),$$

or by direct computation:

$$I(S_1;S_2)=\sum_{j_1(j_1=j_2)}\Pr(j_1)\log_2\frac{\Pr(j_1)}{\Pr(j_1)\Pr(j_1)}=\sum_{j_1(j_1=j_2)}\Pr(j_1)\log_2\frac{1}{\Pr(j_1)}=H(S_1)$$

Problem 2.7

a) $\left|\mathbf{C}_{sg}\right|=\sigma_s{}^2\sigma_s{}^2-\mu_{sg}{}^2 \qquad \mathbf{C}_{sg}{}^{-1}=\frac{1}{\left|\mathbf{C}_{sg}\right|}\cdot\begin{bmatrix}\sigma_g{}^2 & -\mu_{sg}\\ -\mu_{sg} & \sigma_s{}^2\end{bmatrix}$

$$p_{sg}(x,y)=\frac{1}{\sqrt{(2\pi)^2\left|\mathbf{C}_{sg}\right|}}\mathrm{e}^{-\frac{1}{2}\frac{\sigma_g{}^2(x-m_s)^2+\sigma_s{}^2(y-m_g)^2-2\mu_{sg}(x-m_s)(y-m_g)}{\sigma_s{}^2\sigma_g{}^2-\mu_{sg}{}^2}}$$

$\mu_{sg}=0$: (in the sequel, assumption of zero-mean signals, $m_s=m_g=0$)

$$p_{sg}(x,y)=\frac{1}{\sqrt{(2\pi)^2\sigma_s{}^2\sigma_g{}^2}}\mathrm{e}^{-\frac{1}{2}\frac{\sigma_g{}^2x^2+\sigma_s{}^2y^2}{\sigma_s{}^2\sigma_g{}^2}}=\frac{1}{\sqrt{2\pi\sigma_s{}^2}}\cdot\frac{1}{\sqrt{2\pi\sigma_g{}^2}}\cdot\mathrm{e}^{-\frac{1}{2}\frac{x^2}{\sigma_s{}^2}}\cdot\mathrm{e}^{-\frac{1}{2}\frac{y^2}{\sigma_g{}^2}}=p_s(x)\cdot p_g(y)$$

$x=y$: As joint values with $x\neq y$ cannot occur, the following condition holds:

$$\int_{-\infty}^{y-\varepsilon}p_{sg}(x,y)\mathrm{d}x=\int_{y+\varepsilon}^{\infty}p_{sg}(x,y)\,\mathrm{d}\,x=0 \text{ for all } \varepsilon>0 \text{ and } \int_{-\infty}^{\infty}p_{sg}(x,y)\,\mathrm{d}\,y=p_s(x)$$

Using the "sifting property" of $\delta(\cdot)$, these conditions are fulfilled by the following function using the Dirac impulse: $p_{sg}(x,y)=p_s(x)\delta(y-x)$. This also guarantees that the volume below the joint PDF is one:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}p_{sg}(x,y)\mathrm{d}y\mathrm{d}x=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}p_s(x)\delta(y-x)\mathrm{d}y\mathrm{d}x=\int_{-\infty}^{\infty}p_s(x)\underbrace{\int_{-\infty}^{\infty}\delta(y-x)\mathrm{d}y}_{=1}\mathrm{d}x=\int_{-\infty}^{\infty}p_s(x)\mathrm{d}x=1$$

b) $p_{g|s}(y\,|\,x)=\frac{p_{sg}(x,y)}{p_s(x)}=\frac{1}{\sqrt{(2\pi)^2\left(\sigma_s{}^2\sigma_g{}^2-\mu_{sg}{}^2\right)}}\mathrm{e}^{-\frac{1}{2}\frac{\sigma_g{}^2x^2+\sigma_s{}^2y^2-2\mu_{sg}xy}{\sigma_s{}^2\sigma_g{}^2-\mu_{sg}{}^2}}\cdot\sqrt{2\pi\sigma_s{}^2}\,\mathrm{e}^{\frac{1}{2}\frac{x^2}{\sigma_s{}^2}}$

Expressed by normalized covariance coefficients $\rho_{sg}=\mu_{sg}/(\sigma_s\sigma_g)$:

$$p_{g|s}(y\,|\,x)=\frac{\sqrt{2\pi\sigma_s{}^2}}{\sqrt{(2\pi)^2\sigma_s{}^2\sigma_g{}^2\left(1-\mu_{sg}{}^2\right)}}e^{-\frac{1}{2}\left(\frac{\sigma_g{}^2x^2+\sigma_s{}^2y^2-2\sigma_s\sigma_g\mu_{sg}xy}{\sigma_s{}^2\sigma_g{}^2\left(1-\rho_{sg}{}^2\right)}-\frac{x^2}{\sigma_s{}^2}\right)}$$

$$=\frac{1}{\sqrt{2\pi\sigma_g{}^2\left(1-\rho_{sg}{}^2\right)}}\mathrm{e}^{-\frac{1}{2}\underbrace{\frac{\sigma_g{}^2x^2+\sigma_s{}^2y^2-2\sigma_s\sigma_g\rho_{sg}xy-x^2\sigma_g{}^2\left(1-\rho_{sg}{}^2\right)}{\sigma_s{}^2\sigma_g{}^2\left(1-\rho_{sg}{}^2\right)}}_{=\frac{\sigma_s{}^2y^2-2\sigma_s\sigma_g\rho_{sg}xy+x^2\sigma_g{}^2\rho_{sg}{}^2}{\sigma_s{}^2\sigma_g{}^2\left(1-\rho_{sg}{}^2\right)}}}$$

$$=\frac{1}{\sqrt{2\pi\sigma_g{}^2\left(1-\rho_{sg}{}^2\right)}}\mathrm{e}^{-\frac{1}{2}\frac{\left(\sigma_sy-\sigma_g\rho_{sg}x\right)^2}{\sigma_s{}^2\sigma_g{}^2\left(1-\rho_{sg}{}^2\right)}}=\frac{1}{\sqrt{2\pi\sigma_g{}^2\left(1-\rho_{sg}{}^2\right)}}\mathrm{e}^{-\frac{1}{2}\cdot\frac{\left(y-\frac{\sigma_g}{\sigma_s}\rho_{sg}x\right)^2}{\sigma_g{}^2\left(1-\rho_{sg}{}^2\right)}}$$

Note: this is also the PDF of the prediction error, when $g(t)$ is predicted from $s(t)$ using a predictor coefficient $\sigma_g\rho_{sg}/\sigma_s=\mu_{sg}/\sigma_s{}^2$; the variance of the prediction error is $\sigma_g{}^2(1-\rho_{sg})$.

For uncorrelated signals, $\rho_{sg}=0$:

$$p_{g|s}(y\,|\,x)=\frac{1}{\sqrt{2\pi\sigma_g{}^2}}\mathrm{e}^{-\frac{1}{2}\frac{y^2}{\sigma_g{}^2}}=p_g(y)$$

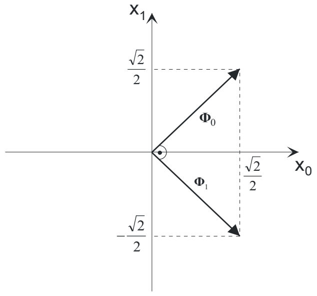
For fully dependent signals, $\rho_{sg}=1$ and $p_{g|s}(y|x)=\delta(y-x)$.

Problem 2.8

a) $\begin{bmatrix}1 & \rho\\ \rho & 1\end{bmatrix}\cdot\begin{bmatrix}\varphi_{k,0}\\ \varphi_{k,1}\end{bmatrix}=\lambda_u\cdot\begin{bmatrix}\varphi_{k,0}\\ \varphi_{k,1}\end{bmatrix} \quad ; \quad k=0,1$

$k=0$: $\phi_{0,0}+\rho\cdot\phi_{0,1}=(1+\rho)\cdot\phi_{0,0} \quad ; \quad \rho\cdot\phi_{0,0}+\phi_{0,1}=(1+\rho)\cdot\phi_{0,1} \quad \Rightarrow \quad \phi_{0,0}=\phi_{0,1}$

k=1 : $\phi_{1,0} + \rho \cdot \phi_{1,1} = (1-\rho)\cdot \phi_{1,0}$; $\rho \cdot \phi_{1,0} + \phi_{1,1} = (1-\rho)\cdot \phi_{1,1} \quad \Rightarrow \quad \phi_{1,0} = -\phi_{1,1}$



Orthonormality:

$$\phi_{0,0}{}^2 + \phi_{0,1}{}^2 = 1 \quad ; \quad \phi_{1,0}{}^2 + \phi_{1,1}{}^2 = 1 \quad \Rightarrow \quad \phi_{0,0}{}^2 = \phi_{0,1}{}^2 = \phi_{1,0}{}^2 = \phi_{1,1}{}^2 = \frac{1}{2} \; \phi_{0,0} = \phi_{0,1} = \pm\frac{\sqrt{2}}{2} \quad ; \quad \phi_{1,0} = -\phi_{1,1} = \pm\frac{\sqrt{2}}{2}$$

b) 4 possible solutions (combinations of ± signs) are (the Figure above shows the case with both signs positive)

$$\mathbf{\Phi}_0 = \pm\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} ; \mathbf{\Phi}_1 = \pm\begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}$$

c) Basis system is orthonormal and real: $\mathbf{\Psi}^{-1}=\mathbf{\Psi}^{T}$

$$\mathbf{\Psi} = \begin{bmatrix} \mathbf{\Phi}_0 & \mathbf{\Phi}_1 \end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \mathbf{\Psi}^{-1}$$

d) $|\mathbf{R}|=1-\rho^2$; $\lambda_1\cdot\lambda_2=(1+\rho)\cdot(1-\rho)=1-\rho^2=|\mathbf{\Lambda}|$ in the eigenvalue system

$$\mathbf{R}\cdot\mathbf{\Psi} = \mathbf{\Psi}\cdot\mathbf{\Lambda} \quad ; \quad \mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

Problem 2.9

Proof of autocorrelation properties:

$$s_{\mathrm{AR}}(n+1) = \rho s_{\mathrm{AR}}(n) + v(n+1)$$

$$s_{\mathrm{AR}}(n+2) = \rho s_{\mathrm{AR}}(n+1) + v(n+2) = \rho^2 s_{\mathrm{AR}}(n) + \rho v(n+1) + v(n+2)$$

$$s_{\mathrm{AR}}(n+k) = \rho s_{\mathrm{AR}}(n+k-1) + v(n+k) = \rho^k s_{\mathrm{AR}}(n) + \sum_{l=1}^{k}\rho^{k-l}v(n+l)$$

$$\varphi_{ss}(k) = \mathcal{E}\{s_{\mathrm{AR}}(n)s_{\mathrm{AR}}(n+k)\}$$

$$= \rho^k \underbrace{\mathcal{E}\{s_{\mathrm{AR}}(n)s_{\mathrm{AR}}(n)\}}_{=\sigma_s{}^2} + \underbrace{\mathcal{E}\left\{s_{\mathrm{AR}}(n)\sum_{l=1}^{k}\rho^{k-l}v(n+l)\right\}}_{=0}$$

$$\varphi_{ss}(-k) = \varphi_{ss}(k) \Rightarrow \varphi_{ss}(k) = \sigma_s{}^2\rho^{|k|}$$

Proof of variance properties:

$$\sigma_v{}^2 = \mathcal{E}\{v^2(n)\} = \mathcal{E}\left\{\left(s_{\mathrm{AR}}(n) - \rho s_{\mathrm{AR}}(n-1)\right)^2\right\}$$

$$= \underbrace{\mathcal{E}\left\{\left(s_{\mathrm{AR}}(n)\right)^2\right\}}_{=\sigma_s{}^2} - 2\rho\underbrace{\mathcal{E}\{s_{\mathrm{AR}}(n)s_{\mathrm{AR}}(n-1)\}}_{=\rho\sigma_s{}^2} + \rho^2\underbrace{\mathcal{E}\left\{\left(s_{\mathrm{AR}}(n-1)\right)^2\right\}}_{=\sigma_s{}^2}$$

$$= \sigma_s{}^2\left(1-\rho^2\right)$$

Proof of spectral properties:

$$B(f) = \frac{1}{1-\rho\mathrm{e}^{-\mathrm{j}2\pi f}}$$

$$\Phi_{ss}(f) = \sigma_v{}^2\left|B(f)\right|^2 = \frac{\sigma_v{}^2}{\left(1-\rho\mathrm{e}^{-\mathrm{j}2\pi f}\right)\rho\mathrm{e}^{\mathrm{j}2\pi f}}$$

$$= \frac{\sigma_v{}^2}{1-\rho\left(\mathrm{e}^{-\mathrm{j}2\pi f}+\mathrm{e}^{\mathrm{j}2\pi f}\right)+\rho^2} = \frac{\sigma_v{}^2}{1-2\rho\cos 2\pi f+\rho^2}$$

alternatively by Fourier transform of ACF:

$$\Phi_{ss}(f)=\sum_{k=-\infty}^{\infty}\varphi_{ss}(k)\mathrm{e}^{-\mathrm{j}2\pi f}=\sigma_s^2\sum_{k=-\infty}^{\infty}\rho^{|k|}\,\mathrm{e}^{-\mathrm{j}2\pi f}=\sigma_s^2\left[\sum_{k=0}^{\infty}\left(\rho\mathrm{e}^{-\mathrm{j}2\pi f}\right)^k+\sum_{k=1}^{\infty}\left(\rho\mathrm{e}^{\mathrm{j}2\pi f}\right)^k\right].$$

With $\sum_{k=0}^{\infty}a^k=\frac{1}{1-a}$ when $|a|<1$:

$$\Rightarrow\Phi_{ss}(f)=\sigma_s^2\left[\frac{1}{1-\rho\mathrm{e}^{-\mathrm{j}2\pi f}}+\frac{1}{1-\rho\mathrm{e}^{\mathrm{j}2\pi f}}-1\right]=\frac{\sigma_s^2\left(1-\rho^2\right)}{1-\rho\left(\mathrm{e}^{-\mathrm{j}2\pi f}+\mathrm{e}^{\mathrm{j}2\pi f}\right)+\rho^2}=\frac{\sigma_v^2}{1-2\rho\cos 2\pi f+\rho^2}$$

Problem 2.10

$$\sigma_s^2\begin{bmatrix}\rho\\ \rho^2\end{bmatrix}=\sigma_s^2\begin{bmatrix}1&\rho\\ \rho&1\end{bmatrix}\begin{bmatrix}a(1)\\ a(2)\end{bmatrix}$$

$$a(1)+\rho a(2)=\rho \qquad \Rightarrow a(1)=\rho\left[1-a(2)\right]$$

$$\rho a(1)+a(2)=\rho^2\Rightarrow\rho^2\left[1-a(2)\right]+a(2)=\rho^2\Rightarrow a(2)\left[1-\rho^2\right]=0$$

For $\rho\neq1$: $a(1)=\rho$; $a(2)=0$. For $\rho=1$: $a(1)+a(2)=1$ (degenerate case)

Problem 2.11

$$\sigma_e^2=\mathcal{E}\left\{e^2(\mathbf{n})\right\}=\mathcal{E}\left\{\left[s(n_1,n_2)-\hat{s}(n_1,n_2)\right]^2\right\};\ \hat{s}(n_1,n_2)=0.5s(n_1-1,n_2)+0.5s(n_1,n_2-1)$$

$$\begin{aligned}\sigma_e^2&=\mathcal{E}\left\{s^2(n_1,n_2)\right\}-(2\cdot0.5)\mathcal{E}\left\{s(n_1,n_2)s(n_1-1,n_2)\right\}-(2\cdot0.5)\mathcal{E}\left\{s(n_1,n_2)s(n_1,n_2-1)\right\}\\&\quad+(0.5\cdot0.5)\mathcal{E}\left\{s^2(n_1-1,n_2)\right\}+(0.5\cdot0.5)\mathcal{E}\left\{s^2(n_1,n_2-1)\right\}+(2\cdot0.25)\mathcal{E}\left\{s(n_1-1,n_2)s(n_1,n_2-1)\right\}\\&=\sigma_s^2-\rho_1\sigma_s^2-\rho_2\sigma_s^2+0.25\sigma_s^2+0.25\sigma_s^2+0.5\rho_1\rho_2\sigma_s^2=\sigma_s^2\left[1.5-\rho_1-\rho_2+0.5\rho_1\rho_2\right]\end{aligned}$$

For $\rho_1=\rho_2=0.95$: $\sigma_e^2=0.05125\cdot\sigma_s^2$

Separable Predictor : Prediction error = variance of innovation, according to (2.195)

$\sigma_e^2=\sigma_v^2=\sigma_s^2\left[1-\rho_1^2\right]\left[1-\rho_2^2\right]$. For $\rho_1=\rho_2=0.95$: $\sigma_e^2=0.009506\sigma_s$

Problem 2.12

a) Result as in Problem 4.4:

$\sigma_e^2=\sigma_v^2=\sigma_s^2\left[1-\rho_1^2\right]\left[1-\rho_2^2\right]$. For $\rho_1=\rho_2=0.95$: $\sigma_e^2=0.009506\sigma_s^2$.

b) As a result of translation shift: $s(n_1,n_2,n_3-1)=s(n_1+k_1,n_2+k_2,n_3)$

$$\begin{aligned}\sigma_e^2&=\mathcal{E}\left\{\left[s(n_1,n_2,n_3)-s(n_1,n_2,n_3-1)\right]^2\right\}\approx\mathcal{E}\left\{\left[s(n_1,n_2,n_3)-s(n_1+k_1,n_2+k_2,n_3)\right]^2\right\}\\&=\mathcal{E}\left\{s^2(n_1,n_2,n_3)\right\}-2\mathcal{E}\left\{s(n_1,n_2,n_3)s(n_1+k_1,n_2+k_2,n_3)\right\}+\mathcal{E}\left\{s^2(n_1+k_1,n_2+k_2,n_3)\right\}\\&=2\sigma_s^2\left[1-\rho_1^{|k|}\rho_2^{|l|}\right]\end{aligned}$$

For $\rho_1=\rho_2=0.95$: $\sigma_e^2=0.8025\sigma_s^2$

c) By knowing actual shift for prediction: $x(m-k,n-l,o-1)=x(m,n,o)$

$$\sigma_e^2=\mathcal{E}\left\{\left[s(n_1,n_2,n_3)-s(n_1-k_1,n_2-k_2,n_3-1)\right]^2\right\}=\mathcal{E}\left\{\left[s(n_1,n_2,n_3)-s(n_1,n_2,n_3)\right]^2\right\}=0$$

Problem 2.13

a) $\mathbf{T}^{\text{Haar(4)}}=\frac{1}{2}\begin{bmatrix}1&1&1&1\\1&1&-1&-1\\\sqrt2&-\sqrt2&0&0\\0&0&\sqrt2&-\sqrt2\end{bmatrix}$; $\mathbf{T}^{\text{Walsh(4)}}=\frac{1}{2}\begin{bmatrix}1&1&1&1\\1&1&-1&-1\\1&-1&-1&1\\1&-1&1&-1\end{bmatrix}$

b) $\mathbf{C}=\left[\mathbf{T}_\mathrm{v}\mathbf{S}\right]\mathbf{T}_\mathrm{h}^\mathrm{T}$

<u>i) Haar basis :</u>

Vertical transform step $\mathbf{C}_\mathrm{v}=\mathbf{T}_\mathrm{v}\mathbf{S}$

$$\mathbf{C}_\mathrm{v}=\frac{1}{2}\begin{bmatrix}1&1&1&1\\1&1&-1&-1\\\sqrt2&-\sqrt2&0&0\\0&0&\sqrt2&-\sqrt2\end{bmatrix}\begin{bmatrix}18&4&2&4\\18&4&2&4\\2&4&2&4\\2&4&2&4\end{bmatrix}=\begin{bmatrix}20&8&4&8\\16&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

Horizontal transform step $\mathbf{C}=\mathbf{C}_v\mathbf{T}_h^T$

$$\mathbf{C}=\frac{1}{2}\begin{bmatrix}20&8&4&8\\16&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}1&1&\sqrt{2}&0\\1&1&-\sqrt{2}&0\\1&-1&0&\sqrt{2}\\1&-1&0&-\sqrt{2}\end{bmatrix}=\begin{bmatrix}20&8&6\sqrt{2}&-2\sqrt{2}\\8&8&8\sqrt{2}&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

ii) Walsh basis :

Vertical transform step $\mathbf{C}_v=\mathbf{T}_v\mathbf{S}$

$$\frac{1}{2}\begin{bmatrix}1&1&1&1\\1&1&-1&-1\\1&-1&-1&1\\1&-1&1&-1\end{bmatrix}\begin{bmatrix}18&4&2&4\\18&4&2&4\\2&4&2&4\\2&4&2&4\end{bmatrix}=\begin{bmatrix}20&8&4&8\\16&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

Horizontal transform step $\mathbf{C}=\mathbf{C}_v\mathbf{T}_h^T$

$$\mathbf{C}=\frac{1}{2}\begin{bmatrix}20&8&4&8\\16&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}1&1&1&1\\1&1&-1&-1\\1&-1&-1&1\\1&-1&1&-1\end{bmatrix}=\begin{bmatrix}20&8&8&4\\8&8&8&8\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

c) Haar transform better suitable for this case, as more coefficients are zero, which typically requires less rate for encoding.

Problem 2.14

a) $$\mathbf{T}^{\cos}=\sqrt{\frac{2}{3}}\begin{bmatrix}\frac{\sqrt{2}}{2}&\frac{\sqrt{2}}{2}&\frac{\sqrt{2}}{2}\\\cos\frac{\pi}{6}&\cos\frac{\pi}{2}&-\cos\frac{\pi}{6}\\\cos\frac{\pi}{3}&\cos\pi&\cos\frac{\pi}{3}\end{bmatrix}=\sqrt{\frac{2}{3}}\begin{bmatrix}\frac{\sqrt{2}}{2}&\frac{\sqrt{2}}{2}&\frac{\sqrt{2}}{2}\\\frac{\sqrt{3}}{2}&0&-\frac{\sqrt{3}}{2}\\\frac{1}{2}&-1&\frac{1}{2}\end{bmatrix}=\begin{bmatrix}\mathbf{t}_0^T\\\mathbf{t}_1^T\\\mathbf{t}_2^T\end{bmatrix}$$

$$\mathbf{t}_0^T\mathbf{t}_1=\left(\sqrt{\frac{2}{3}}\right)^2\left[\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}\right]=0\qquad \mathbf{t}_0^T\mathbf{t}_0=\left(\sqrt{\frac{2}{3}}\right)^2\left[\left(\frac{\sqrt{2}}{2}\right)^2+\left(\frac{\sqrt{2}}{2}\right)^2+\left(\frac{\sqrt{2}}{2}\right)^2\right]=\frac{2}{3}\cdot\frac{3}{2}=1$$

$$\mathbf{t}_0^T\mathbf{t}_2=\left(\sqrt{\frac{2}{3}}\right)^2\left[\frac{\sqrt{2}}{2}\cdot\frac{1}{2}-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\cdot\frac{1}{2}\right]=0\quad \mathbf{t}_1^T\mathbf{t}_1=\left(\sqrt{\frac{2}{3}}\right)^2\left[\left(\frac{\sqrt{3}}{2}\right)^2+0+\left(\frac{\sqrt{3}}{2}\right)^2\right]=\frac{2}{3}\cdot\frac{3}{2}=1$$

$$\mathbf{t}_1^T\mathbf{t}_2=\left(\sqrt{\frac{2}{3}}\right)^2\left[\frac{1}{2}\cdot\frac{\sqrt{3}}{2}-\frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right]=0\qquad \mathbf{t}_2^T\mathbf{t}_2=\left(\sqrt{\frac{2}{3}}\right)^2\left[\left(\frac{1}{2}\right)^2+1+\left(\frac{1}{2}\right)^2\right]=\frac{2}{3}\cdot\frac{3}{2}=1$$

b) $$\mathbf{C}_{ss}=\sigma_s^2\begin{bmatrix}1&\rho&\rho^2\\\rho&1&\rho\\\rho^2&\rho&1\end{bmatrix}\quad;\quad \mathbf{C}_{cc}=\underbrace{[\mathbf{T}\mathbf{C}_{ss}]}_{\mathbf{C}'_{cc}}\mathbf{T}^T$$

$$\mathbf{C}'_{cc}=\mathbf{T}\mathbf{C}_{ss}=\sigma_s^2\sqrt{\frac{2}{3}}\begin{bmatrix}\frac{\sqrt{2}}{2}&\frac{\sqrt{2}}{2}&\frac{\sqrt{2}}{2}\\\frac{\sqrt{3}}{2}&0&-\frac{\sqrt{3}}{2}\\\frac{1}{2}&-1&\frac{1}{2}\end{bmatrix}\begin{bmatrix}1&\rho&\rho^2\\\rho&1&\rho\\\rho^2&\rho&1\end{bmatrix}$$

$$=\sigma_s^{\,2}\sqrt{\frac{2}{3}}\begin{bmatrix}\frac{\sqrt{2}}{2}\left[1+\rho+\rho^2\right] & \frac{\sqrt{2}}{2}\left[1+2\rho\right] & \frac{\sqrt{2}}{2}\left[1+\rho+\rho^2\right]\\ \frac{\sqrt{3}}{2}\left[1-\rho^2\right] & 0 & \frac{\sqrt{3}}{2}\left[\rho^2-1\right]\\ \frac{1}{2}\left[1+\rho^2\right]-\rho & \rho-1 & \frac{1}{2}\left[1+\rho^2\right]-\rho\end{bmatrix}$$

$$\mathbf{C_{cc}}=\mathbf{C'_{cc}}\mathbf{T}^{\mathrm{T}}=\frac{2}{3}\sigma_s^{\,2}\begin{bmatrix}\frac{\sqrt{2}}{2}\left[1+\rho+\rho^2\right] & \frac{\sqrt{2}}{2}\left[1+2\rho\right] & \frac{\sqrt{2}}{2}\left[1+\rho+\rho^2\right]\\ \frac{\sqrt{3}}{2}\left[1-\rho^2\right] & 0 & \frac{\sqrt{3}}{2}\left[\rho^2-1\right]\\ \frac{1}{2}\left[1+\rho^2\right]-\rho & \rho-1 & \frac{1}{2}\left[1+\rho^2\right]-\rho\end{bmatrix}\begin{bmatrix}\frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{2} & \frac{1}{2}\\ \frac{\sqrt{2}}{2} & 0 & -1\\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}$$

$$=\frac{2}{3}\sigma_s^{\,2}\begin{bmatrix}\frac{3}{2}+2\rho+\rho^2 & 0 & \frac{\sqrt{2}}{2}\left[\rho^2-\rho\right]\\ 0 & \frac{3}{2}\left[1-\rho^2\right] & 0\\ \frac{\sqrt{2}}{2}\left[\rho^2-\rho\right] & 0 & \frac{3}{2}-2\rho+\frac{\rho^2}{2}\end{bmatrix}$$

c) ρ=0.9 : $\mathbf{C}_{\mathbf{cc},1}=\sigma_s^{\,2}\begin{bmatrix}2.8683 & 0 & -.033\\ 0 & .0975 & 0\\ -.033 & 0 & .0342\end{bmatrix}$

ρ=0.5 : $\mathbf{C}_{\mathbf{cc},2}=\sigma_s^{\,2}\begin{bmatrix}1.8333 & 0 & -.177\\ 0 & .75 & 0\\ -.177 & 0 & .41671\end{bmatrix}$

In the second case, higher correlation is observed between c_0 and c_2; less concentration of energy in low-frequency coefficients.

d) tr($\mathbf{C_{ss}}$) = tr($\mathbf{C}_{\mathbf{cc},1}$) = tr($\mathbf{C}_{\mathbf{cc},2}$) = $3\sigma_s^2$. As the transformation is orthonormal, the total energy and variance of the vector is unchanged.

Problem 2.15

a) $\mathbf{T}=\frac{1}{2}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & 0 & 0\\ 0 & 0 & 1 & -1\end{bmatrix}$

$$\mathbf{C}_{cc}=\frac{\sigma_s^2}{4}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & 0 & 0\\ 0 & 0 & 1 & -1\end{bmatrix}\begin{bmatrix}1 & \rho & \rho^2 & \rho^3\\ \rho & 1 & \rho & \rho^2\\ \rho^2 & \rho & 1 & \rho\\ \rho^3 & \rho^2 & \rho & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & 0\\ 1 & 1 & -1 & 0\\ 1 & -1 & 0 & 1\\ 1 & -1 & 0 & -1\end{bmatrix}$$

b)

$$=\frac{\sigma_s^2}{4}\begin{bmatrix}1+\rho+\rho^2+\rho^3 & 1+2\rho+\rho^2 & 1+2\rho+\rho^2 & 1+\rho+\rho^2+\rho^3\\ 1+\rho-\rho^2-\rho^3 & 1-\rho^2 & \rho^2-1 & \rho^2+\rho^3-1-\rho\\ 1-\rho & \rho-1 & \rho^2-\rho & \rho^3-\rho^2\\ \rho^2-\rho^3 & \rho-\rho^2 & 1-\rho & \rho-1\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & 0\\ 1 & 1 & -1 & 0\\ 1 & -1 & 0 & 1\\ 1 & -1 & 0 & -1\end{bmatrix}$$

$$=\frac{\sigma_s^2}{4}\begin{bmatrix}4+6\rho+4\rho^2+2\rho^3 & 0 & \rho^3-\rho & \rho-\rho^3\\ 0 & 4+2\rho-4\rho^2 & \rho-\rho^3 & \rho-\rho^3\\ \rho^3-\rho & \rho-\rho^3 & 2-2\rho & 2\rho^2-\rho^3-\rho\\ \rho-\rho^3 & \rho-\rho^3 & 2\rho^2-\rho^3-\rho & 2-2\rho\end{bmatrix}$$

c) Positive correlation between coefficient c_1, and $c_2|c_3$: If a signal has a local gradient around the center of the block, this will be measured just at different scale. Extremely small negative correlation between c_2/c_3, which are measuring local changes in signal at different positions. No correlation between c_0 and c_1, because the first is measuring the mean, and a superimposed change around the center of the block is positive or negative with same probability. Some positive correlation between c_0 and c_3, and negative between c_0 and c_2: If a positive gradient is found in the right part of the block as measured by c_3, the mean measured by c_0 will by tendency be higher; this will likewise be the case if a negative gradient is found in the left part of the block as measured by c_2.

Problem 2.16

a)
$$\mathcal{F}(\mathbf{t}_0)=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\mathrm{e}^{-\mathrm{j}2\pi f}=\mathrm{e}^{-\mathrm{j}\pi f}\left[\frac{\sqrt{2}}{2}\mathrm{e}^{\mathrm{j}\pi f}+\frac{\sqrt{2}}{2}\mathrm{e}^{-\mathrm{j}\pi f}\right]=\mathrm{e}^{-\mathrm{j}\pi f}\sqrt{2}\cos\pi f$$
$$\mathcal{F}(\mathbf{t}_1)=\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\mathrm{e}^{-\mathrm{j}2\pi f}=\mathrm{e}^{-\mathrm{j}\pi f}\left[\frac{\sqrt{2}}{2}\mathrm{e}^{\mathrm{j}\pi f}-\frac{\sqrt{2}}{2}\mathrm{e}^{-\mathrm{j}\pi f}\right]=\mathrm{e}^{-\mathrm{j}\pi f}\sqrt{2}\,\mathrm{j}\sin\pi f=\mathrm{e}^{-\mathrm{j}\pi\left(\frac{1}{2}-f\right)}\sqrt{2}\sin\pi f$$

b)
$$\left(\frac{\sqrt{2}}{2}\right)^2+\left(\pm\frac{\sqrt{2}}{2}\right)^2=1\quad;\quad\left(\frac{\sqrt{2}}{2}\right)^2+\frac{\sqrt{2}}{2}\left(-\frac{\sqrt{2}}{2}\right)=0$$

c)
$$\sqrt{2}\cos\pi f=\sqrt{2}\sin\left(\pi\left(\tfrac{1}{2}+f\right)\right)=\sqrt{2}\sin\left(\pi\left(\tfrac{1}{2}-f\right)\right)\ \left[\text{with}\quad\cos\alpha=\sin\left(\alpha+\pi/2\right);\sin\alpha=\sin(\pi-\alpha)\right]$$

d)
$$\left[\sqrt{2}\cos\pi f\right]^2+\left[\sqrt{2}\sin\pi f\right]^2=2\cdot\left[\cos^2\pi f+\sin^2\pi f\right]=2$$

Problem 2.17

a) Arguments of the windowing function:
$$w(0)=\sin\frac{\pi}{8}=A\quad;\quad w(1)=\sin\frac{3\pi}{8}=B\quad;\quad w(2)=\sin\frac{5\pi}{8}=B\quad;\quad w(3)=\sin\frac{7\pi}{8}=A$$
Arguments of the cosine function:
$$h_0'(0)=\cos\left(-\frac{\pi}{8}\right)=B\ ;\ h_0'(1)=\cos\left(\frac{\pi}{8}\right)=B\ ;\ h_0'(2)=\cos\left(\frac{3\pi}{8}\right)=A\ ;\ h_0'(3)=\cos\left(\frac{5\pi}{8}\right)=-A$$
$$h_1'(0)=\cos\left(-\frac{3\pi}{8}\right)=A\ ;\ h_1'(1)=\cos\left(\frac{3\pi}{8}\right)=A\ ;\ h_1'(2)=\cos\left(\frac{9\pi}{8}\right)=-B\ ;\ h_1'(3)=\cos\left(\frac{15\pi}{8}\right)=B$$
Basis vectors:
$$\mathbf{t}_0=\left[AB\quad BB\quad BA\quad -AA\right]^{\mathrm{T}}\quad;\quad\mathbf{t}_1=\left[AA\quad BA\quad -BB\quad AB\right]^{\mathrm{T}}$$

b)
$$\mathbf{t}_0^{\mathrm{T}}\mathbf{t}_1=\mathbf{t}_1^{\mathrm{T}}\mathbf{t}_0=A^3B+B^3A-B^3A-A^3B=0$$
$$\mathbf{t}_0^{\mathrm{T}}\mathbf{t}_0=A^2B^2+B^4+B^2A^2+A^4=\mathbf{t}_1^{\mathrm{T}}\mathbf{t}_1=A^4+B^2A^2+B^4+A^2B^2=\left[A^2+B^2\right]^2$$
$$\left[A^2+B^2\right]^2=\left[\sin^2\frac{\pi}{8}+\cos^2\frac{\pi}{8}\right]^2=1$$

c)
$$\mathcal{F}(\mathbf{t}_0)=AB+B^2\mathrm{e}^{-\mathrm{j}2\pi f}+AB\mathrm{e}^{-\mathrm{j}4\pi f}-A^2\mathrm{e}^{-\mathrm{j}6\pi f}=AB\left[1+\mathrm{e}^{-\mathrm{j}4\pi f}\right]+\mathrm{e}^{-\mathrm{j}2\pi f}-A^2\left[\mathrm{e}^{-\mathrm{j}2\pi f}+\mathrm{e}^{-\mathrm{j}6\pi f}\right]$$
$$=AB\mathrm{e}^{-\mathrm{j}2\pi f}\left[\mathrm{e}^{\mathrm{j}2\pi f}+\mathrm{e}^{-\mathrm{j}2\pi f}\right]+\mathrm{e}^{-\mathrm{j}2\pi f}-A^2\mathrm{e}^{-\mathrm{j}4\pi f}\left[\mathrm{e}^{\mathrm{j}2\pi f}+\mathrm{e}^{-\mathrm{j}2\pi f}\right]$$
$$=2AB\mathrm{e}^{-\mathrm{j}2\pi f}\cos\left(2\pi f\right)+\mathrm{e}^{-\mathrm{j}2\pi f}-2A^2\mathrm{e}^{-\mathrm{j}4\pi f}\cos\left(2\pi f\right)=\mathrm{e}^{-\mathrm{j}2\pi f}\left[1+2\cos\left(2\pi f\right)\left(AB-A^2\mathrm{e}^{-\mathrm{j}2\pi f}\right)\right]$$
$$\mathcal{F}(\mathbf{t}_1)=A^2+AB\mathrm{e}^{-\mathrm{j}2\pi f}-B^2\mathrm{e}^{-\mathrm{j}4\pi f}+AB\mathrm{e}^{-\mathrm{j}6\pi f}=1-B^2\left[1+\mathrm{e}^{-\mathrm{j}4\pi f}\right]+AB\left[\mathrm{e}^{-\mathrm{j}2\pi f}+e^{-j3\Omega}\right]\mathrm{e}^{-\mathrm{j}6\pi f}$$
$$=1-B^2\mathrm{e}^{-\mathrm{j}2\pi f}\left[\mathrm{e}^{\mathrm{j}2\pi f}+\mathrm{e}^{-\mathrm{j}2\pi f}\right]+AB\mathrm{e}^{-\mathrm{j}4\pi f}\left[\mathrm{e}^{\mathrm{j}2\pi f}+\mathrm{e}^{-\mathrm{j}2\pi f}\right]$$
$$=1-2B^2\mathrm{e}^{-\mathrm{j}2\pi f}\cos\left(2\pi f\right)+2AB\mathrm{e}^{-\mathrm{j}4\pi f}\cos\left(2\pi f\right)=1-2\mathrm{e}^{-\mathrm{j}2\pi f}\cos\left(2\pi f\right)\left(AB\mathrm{e}^{-\mathrm{j}2\pi f}-B^2\right)$$

d) The basis functions do not express impulse responses of linear-phase filters. However, fast algorithms can be realized, as same factors (A^2,B^2,AB) are used multiple times. By appropriate realization, at most three multiplications/sample must be applied to compute the 1D transform.

Problem 2.18

a) Relationships of filter impulse responses: $h_1(n)=(-1)^{1-n}h_0(1-n)$

Filter with 6 coefficients, lowpass:

$h_0(-2)$	$h_0(-1)$	$h_0(0)$	$h_0(1)$	$h_0(2)$	$h_0(3)$
A	B	C	C	B	A

Filter with 6 coefficients, highpass:

$h_1(-2)=-h_0(3)$	$h_1(-1)=h_0(2)$	$h_1(0)=-h_0(1)$	$h_1(1)=h_0(0)$	$h_1(2)=-h_0(-1)$	$h_1(3)=h_0(-2)$
–A	B	–C	C	–B	A

$$\sum_{n=-2}^{3}h_0(n)h_1(n)=-A^2+B^2-C^2+C^2-B^2+A^2=0\text{ and }\sum_{n=-2}^{3}h_k{}^2(n)=2\left(A^2+B^2+C^2\right)>0\quad;\quad k=0,1$$

Filter with 5 coefficients, lowpass:

$h_0(-2)$	$h_0(-1)$	$h_0(0)$	$h_0(1)$	$h_0(2)$	$h_0(3)$
A	B	C	B	A	0

Filter with 5 coefficients, highpass:

$h_1(-2)=-h_0(3)$	$h_1(-1)=h_0(2)$	$h_1(0)=-h_0(1)$	$h_1(1)=h_0(0)$	$h_1(2)=-h_0(-1)$	$h_1(3)=h_0(-2)$
0	A	–B	C	–B	A

$$\sum_{n=-2}^{3} h_0(n)h_1(n) = AB - BC + BC - AB = 0 \text{ and } \sum_{n=-2}^{3} h_k^{\,2}(n) = 2\cdot\left(A^2+B^2\right)+C^2 > 0 \quad ; \quad k=0,1$$

b)

Filter with 6 coefficients:

z-transfer functions of the equivalent polyphase filter components (without subsampling):

$$H'_{0,\mathrm{A}}(z) = H_0(z) = Az^2 + Bz + C + Cz^{-1} + Bz^{-2} + Az^{-3}$$
$$H'_{1,\mathrm{A}}(z) = H_1(z) = -Az^2 + Bz - C + Cz^{-1} - Bz^{-2} + Az^{-3}$$
$$H'_{0,\mathrm{B}}(z) = z^{-1}H_0(z) = Az + B + Cz^{-1} + Cz^{-2} + Bz^{-3} + Az^{-4}$$
$$H'_{1,\mathrm{B}}(z) = z^{-1}H_1(z) = -Az + B - Cz^{-1} + Cz^{-2} - Bz^{-3} + Az^{-4}$$

Subsampling eliminates the odd-indexed filter coefficients:

$$H_{0,\mathrm{A}}(z) = Az + C + Bz^{-1}$$
$$H_{1,\mathrm{A}}(z) = -Az - C - Bz^{-1} = -H_{0,\mathrm{A}}(z)$$
$$H_{0,\mathrm{B}}(z) = B + Cz^{-1} + Az^{-2}$$
$$H_{1,\mathrm{B}}(z) = B + Cz^{-1} + Az^{-2} = H_{0,\mathrm{B}}(z)$$

Due to $H_{1,\mathrm{A}}(z)= -H_{0,\mathrm{A}}(z)$ and $H_{1,\mathrm{B}}(z)=H_{0,\mathrm{B}}(z)$, the A and B branches each require a filter with 2 delay taps and 3 multiplications. As this is performed in parallel for two samples, only 3 multiplications/sample are necessary effectively (instead of 12, when lowpass and highpass filters are applied directly before the subsampling steps).

Filter with 5 coefficients:

z-transfer functions of the equivalent polyphase filter components (without subsampling):

$$H'_{0,\mathrm{A}}(z) = H_0(z) = Az^2 + Bz + C + Bz^{-1} + Az^{-2}$$
$$H'_{1,\mathrm{A}}(z) = H_1(z) = Az^1 - B + Cz^{-1} - Bz^{-2} + Az^{-3}$$
$$H'_{0,\mathrm{B}}(z) = z^{-1}H_0(z) = Az + B + Cz^{-1} + Bz^{-2} + Az^{-3}$$
$$H'_{1,\mathrm{B}}(z) = z^{-1}H_1(z) = A - Bz^{-1} + Cz^{-2} - Bz^{-3} + Az^{-4}$$

After subsampling:

$$H_{0,\mathrm{A}}(z) = Az + C + Az^{-1} = C + A\left(z + z^{-1}\right)$$
$$H_{1,\mathrm{A}}(z) = -B - Bz^{-1} = -B\left(1 + z^{-1}\right)$$
$$H_{0,\mathrm{B}}(z) = B + Bz^{-1} = B\left(1 + z^{-1}\right) = -H_{1,\mathrm{A}}(z)$$
$$H_{1,\mathrm{B}}(z) = A + Cz^{-1} + Az^{-2} = Cz^{-1} + A\left(1 + z^{-2}\right) = z^{-1}H_{0,\mathrm{A}}(z)$$

Here, again 3 multiplications/sample are necessary (instead of 10 for direct computation).
In both the A and B branches, one filter with one and one with two delay taps must be implemented.

Problem 2.19

a) $$\Phi_{ss}(f=1/2) = \frac{\sigma_s^{\,2}(1-\rho^2)}{1+2\rho+\rho^2} = \frac{\sigma_s^{\,2}(1-\rho)}{1+\rho} = \frac{\sigma_s^{\,2}}{9} \Rightarrow 1+\rho = 9(1-\rho) \Rightarrow \rho = 0.8$$
$$\sigma_v^{\,2} = \sigma_s^{\,2}(1-\rho^2) = 0.36\sigma_s^{\,2}$$

b) Prediction of $s(n)$ is performed from $s(n–2)$. Hence,
$$\varphi_{ss}(2) = \sigma_s^{\,2}\rho^2 \Rightarrow a_{opt} = \frac{\varphi_{ss}(2)}{\sigma_s^{\,2}} = \rho^2$$

c) $$e_{\mathrm{e}}(n') = s(n') - \rho^2 s(n'-1), \quad n' = \lfloor n/2 \rfloor$$
$$\sigma_{e_e}^{\,2} = \mathcal{E}\left\{\left(s(n) - \rho^2 s(n-2)\right)^2\right\} = \underbrace{\mathcal{E}\left\{s^2(n)\right\}}_{\sigma_s^2} - 2\rho^2\underbrace{\mathcal{E}\left\{s(n)s(n-2)\right\}}_{\sigma_s^2\rho^2} + \rho^4\underbrace{\mathcal{E}\left\{s^2(n-2)\right\}}_{\sigma_s^2} = \sigma_s^{\,2}(1-\rho^4) \quad \Rightarrow G = \frac{1}{1-\rho^4}$$

d) $e_e(n') = s(2n') - \rho^2 s(2n'-2); \quad e_o(n') = s(2n'-1) - \rho^2 s(2n'-3)$

$$\mathcal{E}\{e_e(n')e_o(n')\} = \mathcal{E}\{[s(n)-\rho^2 s(n-2)][s(n-1)-\rho^2 s(n-3)]\} = \underbrace{\mathcal{E}\{s(n)s(n-1)\}}_{\sigma_s^2\rho} - \rho^2 \underbrace{\mathcal{E}\{s(n-1)s(n-2)\}}_{\sigma_s^2\rho}$$

$$-\rho^2 \underbrace{\mathcal{E}\{s(n)s(n-3)\}}_{\sigma_s^2\rho^3} + \rho^4 \underbrace{\mathcal{E}\{s(n-2)s(n-3)\}}_{\sigma_s^2\rho} = \sigma_s^2\left(\rho-\rho^3\right)$$

Consequently, the prediction errors within the two polyphase components are *not* uncorrelated.

Problem 2.20

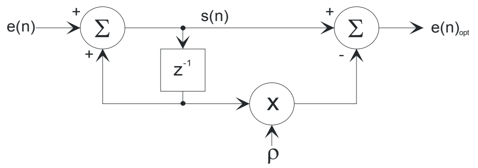
a) $\sigma_s^2 = \dfrac{\sigma_v^2}{\left(1-\rho^2\right)} = \dfrac{7}{\left(1-\frac{9}{16}\right)} = 16$

b) $$\sigma_e^2 = \mathcal{E}\left\{\left(s(n)-s(n-1)\right)^2\right\}$$
$$= \mathcal{E}\left\{s^2(n)\right\} - 2\mathcal{E}\left\{s(n)s(n-1)\right\} + \mathcal{E}\left\{s^2(n-1)\right\} = 2\sigma_s^2\left(1-\rho\right) = 8 \qquad G_{\text{real}} = \frac{\sigma_s^2}{\sigma_e^2} = \frac{16}{8}$$

c) $G_{\text{opt}} = \dfrac{\sigma_s^2}{\sigma_v^2} = \dfrac{16}{7} \quad ; \quad \dfrac{G_{\text{real}}}{G_{\text{opt}}} = \dfrac{8}{7}$

d) $$\Phi_{ee}(f) = \Phi_{ss}(f)\left|A(f)\right|^2 = \frac{\sigma_v^2}{1-2\rho\cos(2\pi f)+\rho^2}\left(1-\mathrm{e}^{\mathrm{j}2\pi f}\right)\left(1-\mathrm{e}^{-\mathrm{j}2\pi f}\right)$$
$$= \frac{2\sigma_v^2\left(1-\cos(2\pi f)\right)}{1-2\rho\cos(2\pi f)+\rho^2} = \frac{14\cdot\left(1-\cos(2\pi f)\right)}{\frac{25}{16}-\frac{3}{2}\cos(2\pi f)}$$

e) $B(z) = \dfrac{1-\rho z^{-1}}{1-z^{-1}}$

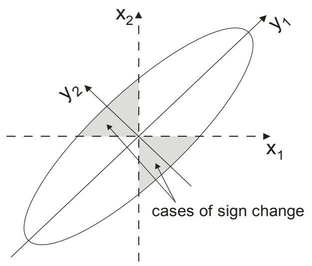


Problem 2.21

a) $s(n)$ is zero-mean Gaussian AR(1) with variance $\sigma_s^2=\sigma_v^2/(1-\rho^2)$. Therefore,

$$\Pr[b(n)=0] = \frac{1}{\sqrt{2\pi\sigma_s^2}}\int_{-\infty}^{C} e^{-\frac{x^2}{2\sigma_s^2}}\,\mathrm{d}x \ \text{ and } \Pr[b(n)=1] = 1-\Pr[b(n)=0].$$

b) The probability of a state transition follows from the probability that the sign of $s(n+1)$ is different from the sign of $s(n)$. The joint probability density of the two samples can be modelled as a 2D Gaussian with zero mean, cf. (2.151). With random variables x_1 referring $s(n)$, and x_2 referring $s(n+1)$, cases of sign change can be found in the gray shaded areas of the elliptically shaped pdf in the following figure:



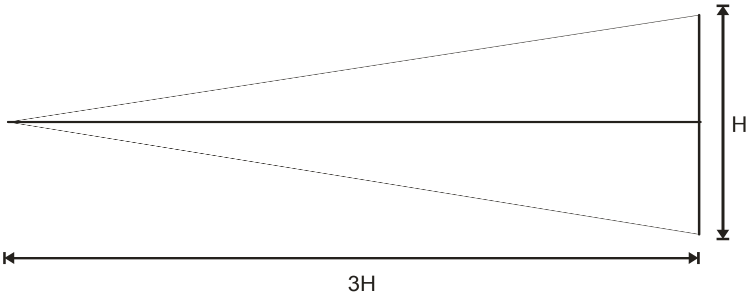
Similar to the transformation in (2.145)ff., the problem can better be solved in the sum/difference coordinate system y_1/y_2. Due to the symmetry over all quadrants, the probability of a sign change can be expressed as

$$\Pr[\operatorname{sgn}(s(n+1)) \neq \operatorname{sgn}(s(n))] = \frac{4}{4\pi\sigma_s^2\sqrt{(1-\rho^2)}}\int_0^{\infty}\int_0^{y_1} e^{-\frac{y_1^2(1-\rho)+y_2^2(1+\rho)}{4\sigma_s^2(1-\rho^2)}}\,\mathrm{d}y_2\,\mathrm{d}y_1.$$

Since after thresholding with C=0, Pr(0)=Pr(1), the probability of sign change can directly be used as Pr(0|1)=Pr(1|0) in the Markov process.

Problem 3.1

A vertical sinusoidal test pattern is shown on an HD display (N_2=1080 lines) with height H. A viewer is observing from a distance D=3H.



a) Total viewing angle: $2\arctan\left((H/2)/3H\right)=2\arctan\left(1/6\right)\approx 18.92^o$. There are approximatlely 5·18,62=94,62 periods visible on the screen, the wavelength is 0.0105H.

b) H corresponds to 1080 sampling units, one period is approx. 0.0105·1080≈11.41 samples.

c) Yes, since the viewing angle would be half and the number of cycles/degree would be doubled

d) (2.258) is type-II DCT, where the basis vector $\mathbf{t}_1$ represents a sampled version of half a cosine period. With the result from b), the block length should be approx. 23 (precisely 22.82) samples.

e) With 3H viewing distance, one cycle (two lines) relates to 18.92/540≈0.030537 degrees, therefore the spatial frequency is 28.54 cy/deg. The number of cy/deg scales linearly with the viewing distance, therefore it would be visible as 60 cy/deg at a distance od 3H·60/28.54≈6.3H. Assume that the HVS of an observer has an ideal cut-off frequency of 60 *cy/deg*. Beyond this distance, the pattern would be perceived as gray.

Problem 4.1

a)

$$R(D)=\frac{1}{2}\log_2\frac{\sigma_v^{\,2}}{D}=\frac{1}{2}\left[\log_2\sigma_v^{\,2}-\log_2 D\right]$$

$$=\frac{1}{2}\int_{-1/2}^{1/2}\log_2\frac{\Phi_{ss}(f)}{D}\mathrm{d}\,f=\frac{1}{2}\left[\int_{-1/2}^{1/2}\log_2\Phi_{ss}(f)\,\mathrm{d}\,f-\underbrace{\int_{-1/2}^{1/2}\log_2 D\,\mathrm{d}\,f}_{=\log_2 D}\right]$$

$$\Rightarrow\int_{-1/2}^{1/2}\log_2\Phi_{ss}(f)\,\mathrm{d}\,f=\log_2\sigma_v^{\,2}$$

$$\gamma_s^{\,2}=\frac{2^{\left[\int_{-1/2}^{1/2}\log_2\Phi_{ss}(f)\,\mathrm{d}\,f\right]}}{\sigma_s^{\,2}}=\frac{2^{\log_2\sigma_v^{\,2}}}{\sigma_s^{\,2}}=\frac{\sigma_v^{\,2}}{\sigma_s^{\,2}}$$

b)

$$R_G=\underbrace{\frac{1}{2}\log_2\frac{\sigma_s^{\,2}}{D}}_{R(D)\text{ uncorrelated}}-\underbrace{\frac{1}{2}\log_2\frac{\sigma_v^{\,2}}{D}}_{R(D)\text{ correlated}}=\frac{1}{2}\log_2\frac{\sigma_{sx}^{\,2}}{D}-\frac{1}{2}\log_2\frac{\sigma_s^{\,2}\left(1-\rho^2\right)}{D}$$

$$=\frac{1}{2}\log_2\frac{\sigma_s^{\,2}}{D}-\frac{1}{2}\log_2\frac{\sigma_{xs}^{\,2}}{D}-\frac{1}{2}\log_2\left(1-\rho^2\right)=-\frac{1}{2}\log_2\left(1-\rho^2\right)$$

$$\text{2D}:\sigma_v^{\,2}=\sigma_s^{\,2}\left(1-\rho_1^{\,2}\right)\left(1-\rho_2^{\,2}\right)$$

$$\Rightarrow R_{\mathrm{G}}=\frac{1}{2}\log_2\frac{\sigma_s^{\,2}}{D}-\frac{1}{2}\log_2\frac{\sigma_s^{\,2}\left(1-\rho_1^{\,2}\right)\left(1-\rho_2^{\,2}\right)}{D}=-\frac{1}{2}\log_2\left(1-\rho_1^{\,2}\right)-\frac{1}{2}\log_2\left(1-\rho_2^{\,2}\right)$$

$\rho_1=\rho_2=0.95$: $R_{G,1D}$=1.679 bit ; $R_{G,2D}$=1.679+1.679=3.358 bit

c)

$$R(D_{\max})=\frac{1}{2}\log_2\frac{\sigma_s^{\,2}\cdot\left(1-\rho_1^{\,2}\right)\cdot\left(1-\rho_2^{\,2}\right)}{D_{\max}}$$

$$D_{\max}\le\Phi_{ss}(1/2,1/2)=\sigma_s^{\,2}\frac{(1-\rho_1^{\,2})}{1+2\rho_1+\rho_1^{\,2}}\cdot\frac{(1-\rho_2^{\,2})}{1+2\rho_2+\rho_2^{\,2}}$$

$$=\sigma_s^{\,2}\frac{(1-\rho_1)(1+\rho_1)}{(1+\rho_1)^2}\cdot\frac{(1-\rho_2)(1+\rho_2)}{(1+\rho_2)^2}$$

$$=\sigma_s^{\,2}\frac{(1-\rho_1)}{(1+\rho_1)}\cdot\frac{(1-\rho_2)}{(1+\rho_2)}$$

$$\Rightarrow R(D_{\max}) = \frac{1}{2}\log_2 \frac{\sigma_s^{\,2}(1-\rho_1^{\,2})(1-\rho_2^{\,2})}{\sigma_s^{\,2}\frac{(1-\rho_1)}{(1+\rho_1)}\cdot\frac{(1-\rho_2)}{(1+\rho_2)}}$$

$$= \frac{1}{2}\log_2\left[(1+\rho_1)^2\cdot(1+\rho_2)^2\right] = \log_2\left[(1+\rho_1)(1+\rho_2)\right]$$

$\rho_1=\rho_2=0.95$: $R(D_{\max})=1.92$ bit

d) $$\Phi_{ss}(f) = \frac{\sigma_s^{\,2}(1-\rho^2)}{1-2\rho\cos(2\pi f)+\rho^2}$$

$$\Theta\left(f_{\max}=1/8\right) = \Phi_{ss}(1/8) = \frac{\sigma_s^{\,2}(1-\rho^2)}{1-2\rho\frac{\sqrt{2}}{2}+\rho^2} = \sigma_s^{\,2}\frac{1-\rho^2}{1-\sqrt{2}\rho+\rho^2}$$

$$\Theta\left(f_{\max}=1/4\right) = \Phi_{ss}(1/4) = \frac{\sigma_s^{\,2}(1-\rho^2)}{1-2\rho\cdot 0+\rho^2} = \sigma_s^{\,2}\frac{1-\rho^2}{1+\rho^2}$$

$$\Theta\left(f_{\max}=1/2\right) = \Phi_{ss}(1/2) = \frac{\sigma_s^{\,2}(1-\rho^2)}{1+2\rho+\rho^2} = \sigma_s^{\,2}\frac{1-\rho}{1+\rho}$$

Θ/σ_s^2	$f_{\max}=1/8$	$f_{\max}=1/4$	$f_{\max}=1/2$
$\rho=0.5$	1.3815	0.6	0.3333
$\rho=0.95$	0.1744	0.0512	0.0256

Problem 4.2

Entropy: 1.92 bit

Shannon code: $z_1=2$ bit, $z_2=z_3=z_4=3$ bit, $R=2.6$ bit

Huffman code: $z_1=1$ bit, $z_2=2$ bit, $z_3=z_4=3$ bit or $z_1=z_2=z_3=z_4=2$ bit, $R=2$ bit in both cases.

Problem 4.3

a) $H(S) = -\left[0,25\cdot\log_2 0,25 + 0,75\cdot\log_2 0,75\right] = 0,5+0,311 = 0,811$ bit

b) $\Pr(0,0) = 0.25\cdot 0.25 = 0,0625 = 2^{-4}$

$\Pr(0,1) = \Pr(1,0) = 0.25\cdot 0.75 = 0.1875 = 3\cdot 2^{-4}$

$\Pr(1,1) = 0.75\cdot 0.75 = 0.5625 = 9\cdot 2^{-4}$

$\Pr(0,0,0) = 0.25^3 = 0.015625 = 2^{-6}$

$\Pr(0,0,1) = \Pr(0,1,0) = \Pr(1,0,0) = 0.25^2\cdot 0.75 = 0.046875 = 3\cdot 2^{-6}$

$\Pr(0,1,1) = \Pr(1,1,0) = \Pr(1,0,1) = 0.75^2\cdot 0.25 = 0.140625 = 9\cdot 2^{-6}$

$\Pr(1,1,1) = 0.75^3 = 0.421875 = 27\cdot 2^{-6}$

c) $i(j) = -\log_2 \Pr(j)$; z_j allocated number of bits

First-order probability : Encoding of separate source symbols

source symbol	$i(j)$ [bit]	z_j Shannon	z_j Huffman
0	2	2	1
1	0.415	1	1

R_{Shannon}=0.25·2 bit+0.75·1 bit=1.25 bit

R_{Huffman}=0.25·1 bit+0.75·1 bit=1 bit

Joint probability of second order: Encoding of source symbol vectors, K=2

source symbol	$i(j)$ [bit]	z_j Shannon	z_j Huffman
0,0	4	4	3
0,1	2.415	3	3
1,0	2.415	3	2
1,1	0.830	1	1

R_{Shannon}=(0.0625·4 bit+2·0.1875·3 bit+0.5625·1 bit):2 =0.96875 bit = H + 0.1578 bit

R_{Huffman}=(0.0625·3 bit+0.1875·3 bit+0.1875·2 bit +0.5625·1 bit):2=0.84375 bit = H + 0.03275 bit

Joint probability of third order: Encoding of source symbol vectors, K=3:

source symbol	$i(j)$ [bit]	z_j Shannon	z_j Huffman
0,0,0	6	6	5
0,0,1	4.415	5	5
0,1,0	4.415	5	5
1,0,0	4.415	5	5

0,1,1	2.830	3	3
1,0,1	2.830	3	3
1,1,0	2.830	3	3
1,1,1	1.245	2	1

R_{Shannon}=(0.015625·6 bit+3·0.046875·5 bit+3·0.140625·3 bit+0.421875·2 bit):3
=0.96875 bit = H + 0.1578 bit
R_{Huffman}=(0.015625·5 bit+3·0.046875·5 bit+3·0.140625·3 bit+0.421875·1 bit):3
=0.8229 bit = H + 0.0119 bit

d) Using (2.163) :

$$\Pr(0|1)=\Pr(0)\cdot\left[\Pr(0|1)+\Pr(1|0)\right] \quad\Rightarrow \Pr(0|1)\cdot\left[1-\Pr(0)\right]=\Pr(0)\cdot\Pr(1|0)$$

$$\Pr(0|1)=\frac{\Pr(0)\cdot\Pr(1|0)}{\Pr(1)}=\frac{0.25\cdot 0.5}{0.75}=0.166\overline{6}$$

$$\Pr(0|0)=1-\Pr(1|0)=0.5\ ;\ \Pr(1|1)=1-\Pr(0|1)=0.833\overline{3}$$

e) Using (2.135) :

$$\Pr(0,0)=\Pr(0|0)\cdot\Pr(0)=0.5\cdot 0.25=0.125$$

$$\Pr(0,1)=\Pr(0|1)\cdot\Pr(1)=0.166\overline{6}\cdot 0.75=0.125$$

$$\Pr(1,0)=\Pr(1|0)\cdot\Pr(0)=0.5\cdot 0.25=0.125$$

$$\Pr(1,1)=\Pr(1|1)\cdot\Pr(1)=0.833\overline{3}\cdot 0.75=0.625$$

The bit allocation of the Huffman code is identical with the result of c). As however the probabilities are different, the rate is:
R_{Huffman}=(2·0.125·3 bit+0.125·2 bit +0.625·1 bit):2=0.8125 bit

f) Using (2.182)-(2.183):

$$H(b)\big|_{b=1}=0.166\overline{6}\cdot 2.58496+0.833\overline{3}\cdot 0.26303=0.650022\ \text{ bit}$$

$$H(b)\big|_{b=0}=0.5\cdot 1+0.5\cdot 1=1\ \text{ bit}$$

$$H(b)=0.25\cdot 1+0.75\cdot 0.650022=0.73751\ \text{ bit}$$

Problem 4.4

a) $H(\boldsymbol{S})=0.8\log_2 0.8+0.2\log_2 0.2=0.7219$ bit

b) Boundaries of probability intervals:
0["A"]0.8["B"]1
0["AA"]0.64["AB"]0.8["BA"]0.96["BB"]1
0["AAA"]0.512["AAB"]0.64["ABA"]0.768["ABB"]0.8["BAA"]0.928["BAB"]0.96["BBA"]0.992["BBB"]1

c) Boundaries of code intervals, 3 bit accuracy :
0 <u>["0"]</u> 0.5 ["1"] 1
0 ["00"] 0.25 ["01"] 0.5 ["10"] 0.75 <u>["11"]</u> 1
0 ["000"] 0.125 ["001"] 0.25 ["010"] 0.375 ["011"] 0.5 <u>["100"]</u> 0.625 <u>["101"]</u> 0.75 ["110"] 0.875 ["111"] 1

Boundaries of probability intervals, rounded; underlined intervals exactly correspond with code intervals underlined above, which results in the following mapping:
0["A"]0.75<u>["B"]</u>1 "B"→"11"
0["AA"]0.625<u>["AB"]</u>0.75 "AB"→"101"
0<u>["AAA"]</u>0.5<u>["AAB"]</u>0.625 "AAA"→"0" ; "AAB"→"100"

The computation of the mean rate (per source symbol) must be modified as compared to (11.29), since the number of source symbols K_j must be considered individually for each code symbol j:

$$R=\sum_j \Pr(j)\frac{z_j}{K_j}$$

$$\Pr(\text{"B"})=0.2\ ;\ \Pr(\text{"AB"})=0.8\cdot 0.2=0.16$$

$$\Pr(\text{"AAA"})=0.8^3=0.512\ ;\ \Pr(\text{"AAB"})=0.8^2\cdot 0.2=0.128$$

$$\Rightarrow R=\underbrace{0.2\cdot 2\cdot\frac{1}{1}}_{\text{"B"}}+\underbrace{0.16\cdot 3\cdot\frac{1}{2}}_{\text{"AB"}}+\underbrace{0.512\cdot 1\cdot\frac{1}{3}}_{\text{"AAA"}}+\underbrace{0.128\cdot 3\cdot\frac{1}{3}}_{\text{"AAB"}}$$

$$\approx 0.4+0.24+0.17066+0.128=0.93866\ \text{ bit}$$

Problem 4.5

a) Method 1 (Fig. 5.1a): 1-2-4-1-1-1-1-2-3. Method 2 (Fig. 5.1b): 1-0-4-1-1-0-3:

b) $\Pr_0(l)=\Pr(1|0)\cdot\left[\Pr(0|0)\right]^{l-1}$; $\Pr_1(l)=\Pr(0|1)\cdot\left[\Pr(1|1)\right]^{l-1}$

$$\Pr(1)=\frac{\Pr(1|0)}{\Pr(0|1)+\Pr(1|0)}=0.333\overline{3}\ ;\ \Pr(0)=\frac{\Pr(0|1)}{\Pr(0|1)+\Pr(1|0)}=0.666\overline{6}$$

$$\Pr_0(1)=\Pr(1|0)=0.4$$

$$\Pr_0(2)=\Pr(1|0)\cdot\Pr(0|0)=0.4\cdot0.6=0.24$$

$$\Pr_0(3)=\Pr(1|0)\cdot\Pr(0|0)^2=0.4\cdot0.36=0.144$$

$$\Pr_1(1)=\Pr(0|1)=0.8$$

$$\Pr_1(2)=\Pr(0|1)\cdot\Pr(1|1)=0.8\cdot0.2=0.16$$

$$\Pr_1(3)=\Pr(0|1)\cdot\Pr(1|1)^2=0.8\cdot0.04=0.032$$

c) According to Fig. 11.18b, only the run-lengths $\Pr_0(l)$ are encoded, but the special case l=0 must also be regarded. To get a consistent result, it is assumed that any run of "0"-bits starts after a "1"-bit and is terminated by reaching the state "1" again. This gives a probability for a run-length $l > 0$

$$\Pr_0(l)=\Pr(0|1)\cdot\Pr(0|0)^{l-1}\cdot\Pr(1|0).$$

The probability of a run l=0 is

$$\Pr_0(l=0)=\Pr(1|1).$$

It is straightforward to show that all run-length probabilities sum up as unity:

$$\Pr(1|1)+\sum_{l=1}^{\infty}\Pr(0|1)\cdot\Pr(0|0)^{l-1}\cdot\Pr(1|0)=\Pr(1|1)+\Pr(0|1)\cdot\Pr(1|0)\cdot\sum_{l=0}^{\infty}\Pr(0|0)^l$$

$$=\Pr(1|1)+\Pr(0|1)\cdot\Pr(1|0)\cdot\frac{1}{1-\Pr(0|0)}=\Pr(1|1)+\Pr(0|1)\cdot\Pr(1|0)\cdot\frac{1}{\Pr(1|0)}$$

$$=\Pr(1|1)+\Pr(0|1)=1.$$

As all run-lengths l>3 shall be handled by the ESCAPE symbol, for which the probability can be computed as follows:

$$\Pr('l=0')=\Pr(1|1)=0.2$$

$$\Pr('l=1')=\Pr(0|1)\cdot\Pr(1|0)=0.8\cdot0.4=0.32$$

$$\Pr('l=2')=\Pr(l=1)\cdot\Pr(0|0)=0.32\cdot0.6=0.192$$

$$\Pr('l=3')=\Pr(l=2)\cdot\Pr(0|0)=0.192\cdot0.6=0.1152$$

$$\Pr('ESC')=1-\sum_{l=0}^{3}\Pr(l)=1-0.2-0.32-0.192-0.1152=0.1728.$$

A Huffman code design gives: ESCAPE and l=3 : 3 bits ; l=0, l=1 and l=2: 2 bits.

Problem 4.6

a) Training sequence vectors:

$\mathbf{x}(0)=[-1\ 0]^T$; $\mathbf{x}(1)=[-3\ -2]^T$; $\mathbf{x}(2)=[\ 1\ 1]^T$; $\mathbf{x}(3)=[-5\ -4]^T$; $\mathbf{x}(4)=[0\ 1]^T$;
$\mathbf{x}(5)=[1\ 0]^T$; $\mathbf{x}(6)=[2\ 2]^T$

1st iteration: Squared distances according to (4.61)

	$\mathbf{x}(0)$	$\mathbf{x}(1)$	$\mathbf{x}(2)$	$\mathbf{x}(3)$	$\mathbf{x}(4)$	$\mathbf{x}(5)$	$\mathbf{x}(6)$
$\mathbf{y}_0$	<u>1</u>	<u>5</u>	8	<u>25</u>	5	5	18
$\mathbf{y}_1$	5	25	<u>0</u>	61	<u>1</u>	<u>1</u>	<u>2</u>

In partition of $\mathbf{y}_0$: $\mathbf{x}(0)$; $\mathbf{x}(1)$; $\mathbf{x}(3)$

In partition of $\mathbf{y}_1$: $\mathbf{x}(2)$; $\mathbf{x}(4)$; $\mathbf{x}(5)$; $\mathbf{x}(6)$

$$\mathbf{y}^{(1)}{}_{0,\mathrm{opt}}=\left[\frac{-1-3-5}{3}\quad\frac{0-2-4}{3}\right]^T=\left[-3\quad-2\right]^T$$

$$\mathbf{y}^{(1)}{}_{1,\mathrm{opt}}=\left[\frac{1+0+1+2}{4}\quad\frac{1+1+0+2}{4}\right]^T=\left[1\quad1\right]^T$$

2nd iteration: Squared distances according to (4.61)

	$\mathbf{x}(0)$	$\mathbf{x}(1)$	$\mathbf{x}(2)$	$\mathbf{x}(3)$	$\mathbf{x}(4)$	$\mathbf{x}(5)$	$\mathbf{x}(6)$
$\mathbf{y}^{(1)}{}_{0,\mathrm{opt}}$	8	<u>0</u>	25	<u>8</u>	18	20	41
$\mathbf{y}^{(1)}{}_{1,\mathrm{opt}}$	<u>5</u>	25	<u>0</u>	61	<u>1</u>	<u>1</u>	<u>2</u>

In partition of $\mathbf{y}_0$: $\mathbf{x}(1)$; $\mathbf{x}(3)$

In partition of $\mathbf{y}_1$: $\mathbf{x}(0)$; $\mathbf{x}(2)$; $\mathbf{x}(4)$; $\mathbf{x}(5)$; $\mathbf{x}(6)$

$$\mathbf{y}^{(2)}{}_{0,\mathrm{opt}}=\left[\frac{-3-5}{2}\quad\frac{-2-4}{2}\right]^{\mathrm{T}}=\left[-4\quad -3\right]^{\mathrm{T}}$$

$$\mathbf{y}^{(2)}{}_{1,\mathrm{opt}}=\left[\frac{-1+1+0+1+2}{5}\quad\frac{0+1+1+0+2}{5}\right]^{\mathrm{T}}=\left[0.6\quad 0.8\right]^{\mathrm{T}}$$

b) After 1st iteration: Pr(0)=3/7 $\Rightarrow$ i(0)=1.2224 ; Pr(1)=4/7 $\Rightarrow$ i(1)=0.8074
2nd iteration: Distances according to cost function (4.83), λ=0, are identical to result a)
2nd iteration: Distances according to cost function (4.83), λ=65, are given in table below

	x(0)	**x**(1)	**x**(2)	**x**(3)	**x**(4)	**x**(5)	**x**(6)
$\mathbf{y}^{(1)}{}_{0,\mathrm{opt}}$	8+79.5 =87.5	0+79.5 =79.5	25+79.5 =104.5	<u>8+79.5 =87.5</u>	18+79.5 =97.5	20+79.5 =99.5	41+79.5 =120.5
$\mathbf{y}^{(1)}{}_{1,\mathrm{opt}}$	<u>5+52.5 =57.5</u>	<u>25+52.5 =77.5</u>	<u>0+52.5 =52.5</u>	61+52.5 =113.5	<u>1+52.5= 53.5</u>	<u>1+52.5= 53.5</u>	<u>2+52.5 =54.5</u>

Modified allocation to $\mathbf{y}_0$: **x**(3)
Modified allocation to $\mathbf{y}_1$: **x**(0) ; **x**(1) ; **x**(2) ; **x**(4) ; **x**(5) ; **x**(6)

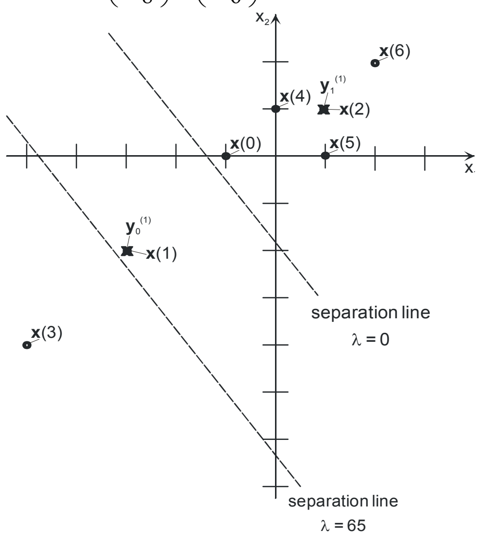
$$\mathbf{y}^{(2)}{}_{0,\mathrm{opt}}=\left[\frac{-5}{1}\quad\frac{-4}{1}\right]^{\mathrm{T}}=\left[-5\quad -4\right]^{\mathrm{T}}$$

$$\mathbf{y}^{(2)}{}_{1,\mathrm{opt}}=\left[\frac{-1-3+1+0+1+2}{6}\quad\frac{0-2+1+1+0+2}{5}\right]^{\mathrm{T}}=\left[0\quad 0.4\right]^{\mathrm{T}}$$

Equation of separation line according to (4.85):

$$2\left[x_1+1\quad x_2+0.5\right]\cdot\begin{bmatrix}4\\3\end{bmatrix}=-\lambda\cdot 0.415\Rightarrow 8x_1+8+6x_2+3=-0.415\lambda$$

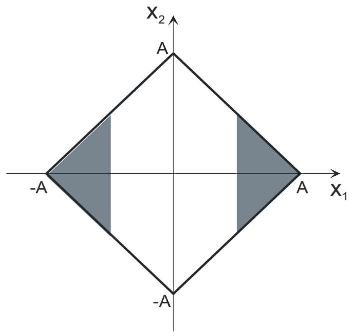
λ=0: $\dfrac{x_1}{\left(-\dfrac{11}{8}\right)}+\dfrac{x_2}{\left(-\dfrac{11}{6}\right)}=1$ $\quad\lambda$=65 : $0.415\lambda\approx 27$: $\dfrac{x_1}{\left(-\dfrac{19}{4}\right)}+\dfrac{x_2}{\left(-\dfrac{19}{3}\right)}=1$

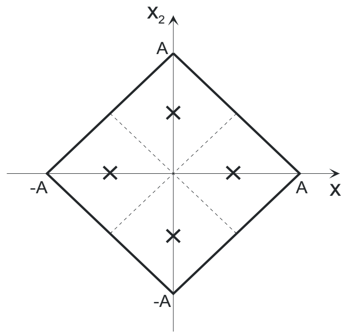


Problem 4.7

a) $$p_{\mathbf{s}}(x_1,x_2)=\begin{cases}\dfrac{1}{2A^2},|x_1|+|x_2|\le A\\[2ex] 0,\ \text{else}\end{cases}$$

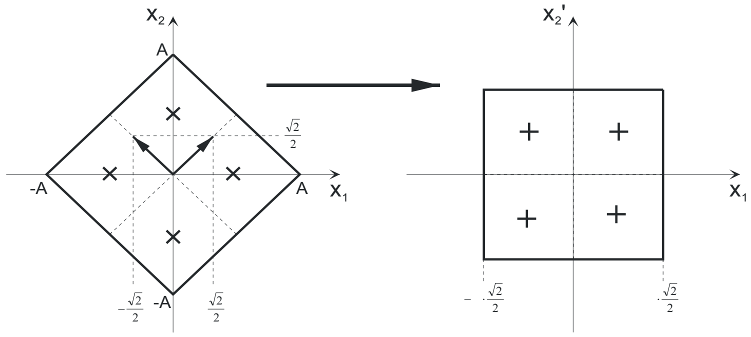
b) Solution graphically from triangular areas $\Pr(|x_1|\ge A/2)=2\cdot\dfrac{1}{2A^2}\cdot\dfrac{1}{2}\cdot A\cdot\dfrac{A}{2}=1/4$





c) Voronoi lines are principal axes of 45°, as all reconstruction values on coordinate axes have same distances from the origin. Optimum is a=A/2 (centroid of Voronoi regions at the centers of the squares, due to uniform PDF)

d) $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ (see Figure)



e) Quantization error identical within all Voronoi regions, can be derived by the $\Delta^2/12$ formula for uniform quantization error variance (related to samples; variance per vector would have double value) :

$$\sigma_q{}^2 = \frac{\left(\sqrt{2}\cdot A/2\right)^2}{12} = \frac{A^2}{24}$$

Problem 5.1

a) 1D : $\sigma_e{}^2 = \sigma_s{}^2\left(1-\rho^2\right) = 4\cdot(1-\frac{3}{4}) = 1 \quad \Rightarrow G = \frac{\sigma_s{}^2}{\sigma_e{}^2} = 4$

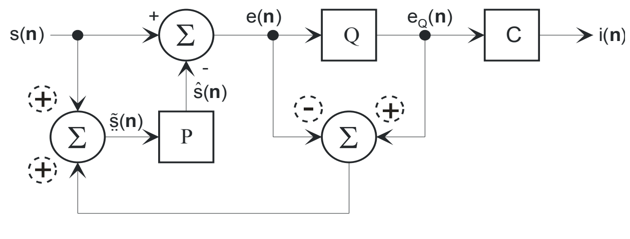
2D : $\sigma_e{}^2 = \sigma_s{}^2\left(1-\rho_1{}^2\right)\left(1-\rho_2{}^2\right) = 4\cdot(1-\frac{3}{4})\cdot(1-\frac{3}{4}) = 0.25 \Rightarrow G = \frac{\sigma_s{}^2}{\sigma_e{}^2} = 16$

b) 1D : $R_G = -\frac{1}{2}\log_2\left(1-\rho^2\right) = -\frac{1}{2}\log_2\left(0.25\right) = 1\text{ bit}$

2D : $$\begin{aligned} R_G &= -\frac{1}{2}\log_2\left(1-\rho_1{}^2\right) - \frac{1}{2}\log_2\left(1-\rho_2{}^2\right) \\ &= 2\cdot\left(-\frac{1}{2}\log_2\left(0.25\right)\right) = 2\text{ bit} \end{aligned}$$

c) At the input of the predictor P, the reconstruction signal $\tilde{s}(\mathbf{n})$ must be available. This can be computed from the original signal $s(\mathbf{n})$ by adding the quantization error:

$$\tilde{s}(\mathbf{n}) = s(\mathbf{n}) - q(\mathbf{n}) = s(\mathbf{n}) - \left[e(\mathbf{n}) - e_Q(\mathbf{n})\right] = s(\mathbf{n}) - e(\mathbf{n}) + e_Q(\mathbf{n})$$



Problem 5.2

With U=2 according to (2.287):

$$\underbrace{\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}}_{\mathbf{T}} \underbrace{\sigma_s^2 \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}}_{\mathbf{C_{ss}}} \underbrace{\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}}_{\mathbf{T}^{\mathrm{T}}} = \frac{1}{2}\sigma_s^2 \begin{bmatrix} 1+\rho & 1+\rho \\ 1-\rho & \rho-1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \underbrace{\sigma_s^2 \begin{bmatrix} 1+\rho & 0 \\ 0 & 1-\rho \end{bmatrix}}_{\mathbf{\Lambda}}$$

$$\mathcal{E}\{c_0^2\} = \sigma_s^2(1+\rho) \quad ; \quad \mathcal{E}\{c_1^2\} = \sigma_s^2(1-\rho)$$

With U=3, using results from Problem 2.14:

$$\mathcal{E}\{c_0^2\} = \sigma_s^2\left(1+\frac{4}{3}\rho+\frac{2}{3}\rho^2\right) \quad ; \quad \mathcal{E}\{c_1^2\} = \sigma_s^2\left(1-\rho^2\right) \quad ; \quad \mathcal{E}\{c_2^2\} = \sigma_s^2\left(1-\frac{4}{3}\rho+\frac{1}{3}\rho^2\right)$$

With separable transform, U_1=V_1=2:

$$\mathcal{E}\{c_{00}^2\} = \sigma_s^2(1+\rho_1)(1+\rho_2) \quad ; \quad \mathcal{E}\{c_{01}^2\} = \sigma_s^2(1+\rho_1)(1-\rho_2)$$

$$\mathcal{E}\{c_{10}^2\} = \sigma_s^2(1-\rho_1)(1+\rho_2) \quad ; \quad \mathcal{E}\{c_{11}^2\} = \sigma_s^2(1-\rho_1)(1-\rho_2)$$

With separable transform, U_1=V_1=3:

$$\mathcal{E}\{c_{00}^2\} = \sigma_s^2\left(1+\frac{4}{3}\rho_1+\frac{2}{3}\rho_1^2\right)\left(1+\frac{4}{3}\rho_2+\frac{2}{3}\rho_2^2\right)$$

$$\mathcal{E}\{c_{01}^2\} = \sigma_s^2\left(1+\frac{4}{3}\rho_1+\frac{2}{3}\rho_1^2\right)\left(1-\rho_2^2\right)$$

$$\mathcal{E}\{c_{02}^2\} = \sigma_s^2\left(1+\frac{4}{3}\rho_1+\frac{2}{3}\rho_2^2\right)\left(1-\frac{4}{3}\rho_2+\frac{1}{3}\rho_2^2\right)$$

$$\mathcal{E}\{c_{10}^2\} = \sigma_s^2\left(1-\rho_1^2\right)\left(1+\frac{4}{3}\rho_2+\frac{2}{3}\rho_2^2\right)$$

$$\mathcal{E}\{c_{11}^2\} = \sigma_s^2\left(1-\rho_1^2\right)\left(1-\rho_2^2\right)$$

$$\mathcal{E}\{c_{12}^2\} = \sigma_s^2\left(1-\rho_1^2\right)\left(1-\frac{4}{3}\rho_2+\frac{1}{3}\rho_2^2\right)$$

$$\mathcal{E}\{c_{20}^2\} = \sigma_s^2\left(1-\frac{4}{3}\rho_1+\frac{1}{3}\rho_1^2\right)\left(1+\frac{4}{3}\rho_2+\frac{2}{3}\rho_2^2\right)$$

$$\mathcal{E}\{c_{21}^2\} = \sigma_s^2\left(1-\frac{4}{3}\rho_1+\frac{1}{3}\rho_1^2\right)\left(1-\rho_2^2\right)$$

$$\mathcal{E}\{c_{22}^2\} = \sigma_s^2\left(1-\frac{4}{3}\rho_1+\frac{1}{3}\rho_1^2\right)\left(1-\frac{4}{3}\rho_2+\frac{1}{3}\rho_2^2\right)$$

Coding gain of discrete transform (12.33)

$$\Rightarrow G_{\mathrm{TC,1D}} = \frac{\frac{1}{U}\sum_{k=0}^{U-1}\mathcal{E}\{c_k^2\}}{\left[\prod_{k=0}^{U-1}\mathcal{E}\{c_k^2\}\right]^{1/U}} \qquad G_{\mathrm{TC,2D}} = \frac{\frac{1}{U_1U_2}\sum_{k_1=0}^{U_1-1}\sum_{k_2=0}^{U_2-1}\mathcal{E}\{c_{k_1k_2}^2\}}{\left[\prod_{k_1=0}^{U_1-1}\prod_{k_2=0}^{U_2-1}\mathcal{E}\{c_{k_1k_2}^2\}\right]^{1/U_1U_2}}$$

e.g. U=2 : $\qquad G_{\mathrm{TC,1D}} = \frac{1}{\sqrt{1-\rho^2}}$

U_1=U_2=2, ρ=ρ_1=ρ_2: $\qquad G_{\mathrm{TC,2D}} = \frac{1}{\sqrt[4]{(1+\rho)^2(1-\rho^2)^2(1-\rho)^2}} = \frac{1}{1-\rho^2}$

Theoretical gain, 1D from (4.33) $\Rightarrow G_{\mathrm{1D}} = \frac{1}{1-\rho^2}$

Theoretical gain, 2D (4.41) $\Rightarrow G_{\mathrm{2D}} = \frac{1}{\left(1-\rho_1^2\right)\cdot\left(1-\rho_2^2\right)}$

	$G_{\mathrm{TC,1D}}$,U=2	$G_{\mathrm{TC,1D}}$,U=3	$G_{\mathrm{TC,2D}}$,U_1=U_2=2	$G_{\mathrm{TC,2D}}$,U_1=U_2=3	G_{1D}	G_{2D}
ρ=0.5	1.1547	1.204	1.3333	1.4497	1.3333	1.7777
ρ=0.95	3.2026	4.7125	10.2564	22.208	10.2564	105.194

<u>Other example :</u>

ρ=0.91, $G_{1D,opt}$=5.8173

DCT with U=2 : $G_{TC,1D}$=2.4119 ; U=3 : $G_{TC,1D}$=3.2253

DCT with U=8 provides the covariance matrix of transform coefficients $\mathbf{C}_{cc}$

$$\mathbf{C}_{cc}=\sigma_s^{\,2}\begin{bmatrix} 6.344 & 0.000 & \mathbf{-0.291} & 0.000 & \mathbf{-0.066} & 0.000 & \mathbf{-0.021} & 0.000 \\ 0.000 & 0.930 & 0.000 & \mathbf{-0.027} & 0.000 & \mathbf{-0.008} & 0.000 & \mathbf{-0.002} \\ \mathbf{-0.291} & 0.000 & 0.312 & 0.000 & \mathbf{-0.001} & 0.000 & \mathbf{0.000} & 0.000 \\ 0.000 & \mathbf{-0.027} & 0.000 & 0.149 & 0.000 & \mathbf{-0.001} & 0.000 & \mathbf{0.000} \\ \mathbf{-0.066} & 0.000 & \mathbf{-0.001} & 0.000 & 0.094 & 0.000 & \mathbf{0.000} & 0.000 \\ 0.000 & \mathbf{-0.008} & 0.000 & \mathbf{-0.001} & 0.000 & 0.068 & 0.000 & \mathbf{0.000} \\ \mathbf{-0.021} & 0.000 & \mathbf{0.000} & 0.000 & \mathbf{0.000} & 0.000 & 0.055 & 0.000 \\ 0.000 & \mathbf{-0.002} & 0.000 & \mathbf{0.000} & 0.000 & \mathbf{0.000} & 0.000 & 0.049 \end{bmatrix}$$

which gives $G_{TC,1D}$=4.6313, and also illustrates that the coefficients are not entirely uncorrelated.

Problem 5.3

a) $$\mathbf{T}^{-1}=\frac{1}{\det\mathbf{T}}\begin{bmatrix}-1 & -1/2\\ -1 & 1/2\end{bmatrix}=\begin{bmatrix}1 & 1/2\\ 1 & -1/2\end{bmatrix}$$

b) Basis vectors have different norms (not unity). Equivalent orthonormal transform is $\mathbf{S}=\frac{\sqrt{2}}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$

c) $$\mathcal{E}\left\{c_0^{\,2}\right\}=\frac{1}{4}\mathcal{E}\left\{\left(s(n)+s(n+1)\right)^2\right\}=\frac{1}{4}\left[\mathcal{E}\left\{s^2(n)\right\}+2\mathcal{E}\left\{s(n)x(n+1)\right\}+\mathcal{E}\left\{s^2(n+1)\right\}\right]=\frac{\sigma_s^{\,2}(1+\rho)}{2}$$

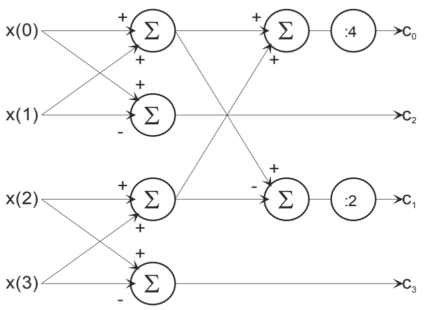
$$\mathcal{E}\left\{c_1^{\,2}\right\}=\mathcal{E}\left\{\left(s(n)-s(n+1)\right)^2\right\}=\left[\mathcal{E}\left\{s^2(n)\right\}-2\mathcal{E}\left\{s(n)s(n+1)\right\}+\mathcal{E}\left\{s^2(n+1)\right\}\right]=2\sigma_s^{\,2}(1-\rho)$$

d) $$\tilde{\mathbf{s}}-\mathbf{s}=\mathbf{T}^{-1}\mathbf{q}=\begin{bmatrix}1 & 1/2\\ 1 & -1/2\end{bmatrix}\begin{bmatrix}q_0\\ q_1\end{bmatrix}=\begin{bmatrix}q_0+q_1/2\\ q_0-q_1/2\end{bmatrix}$$

$$\mathcal{E}\left\{[\tilde{\mathbf{s}}-\mathbf{s}]^{\mathrm{T}}[\tilde{\mathbf{s}}-\mathbf{s}]\right\}=\mathcal{E}\left\{\left(q_0+q_1/2\right)^2\right\}+\mathcal{E}\left\{\left(q_0-q_1/2\right)^2\right\}=2\left[\mathcal{E}\left\{q_0^{\,2}\right\}+\mathcal{E}\left\{\left(\frac{q_1}{2}\right)^2\right\}\right]$$

Consequently, the quantizer step size effecting the quantization error q_0 should be half of that relating to q_1.

e)



$$\mathbf{T}(4)=\begin{bmatrix}1/4 & 1/4 & 1/4 & 1/4\\ 1/2 & 1/2 & -1/2 & -1/2\\ 1 & -1 & 0 & 0\\ 0 & 0 & 1 & -1\end{bmatrix}$$

Problem 6.1

a) Image 1: $\frac{1}{N_1N_2}\sum_{n_1}\sum_{n_2}\left|s(n_1,n_2)-\tilde{s}(n_1,n_2)\right|=\frac{1}{9}\left[|20-19|+|17-18|+\ldots+|14-13|\right]=1$

Image 2 : $\frac{1}{N_1N_2}\sum_{n_1}\sum_{n_2}\left|s(n_1,n_2)-\tilde{s}(n_1,n_2)\right|=\frac{1}{9}\left[|20-20|+|17-17|+\ldots+|14-23|+\ldots+|14-14|\right]=1$

b) Image 1: $\frac{1}{N_1N_2}\sum_{n_1}\sum_{n_2}\left(s(n_1,n_2)-\tilde{s}(n_1,n_2)\right)^2=\frac{1}{9}\left[(20-19)^2+(17-18)^2+\ldots+(14-13)^2\right]=1$

$$\mathrm{PSNR}=10\cdot\log_{10}\frac{255^2}{1}=48.13\ \text{dB}$$

$$\text{Image 2}: \frac{1}{N_1 N_2}\sum_{n_1}\sum_{n_2}\left|s(n_1,n_2)-\tilde{s}(n_1,n_2)\right| = \frac{1}{9}\left[(20-20)^2+(17-17)^2+\ldots+(14-23)^2+\ldots+(14-14)^2\right]=9$$

$$\text{PSNR} = 10\cdot\log_{10}\frac{255^2}{9} = 38.59\ \text{dB}$$

Interpretation: If the squared error (energy of error) criterion is used, single high deviations are penalized more strictly than for the case of errors which are widely dispersed over the image. This phenomenon is in accordance with the visibility of errors (single high errors are clearly visible, not being masked by the image content).

Problem 6.2

a)

i) $\hat{\mathbf{S}} = \begin{bmatrix} 0&0&0&0&0\\ 0&0&1&1&1\\ 0&0&1&1&1\\ 0&0&1&1&1\\ 0&0&0&0&0 \end{bmatrix}$ ii) $\hat{\mathbf{S}} = \begin{bmatrix} 0&0&0&0&0\\ 0&0&0.5&0.5&0.5\\ 0&0.5&1&1&0.5\\ 0&0.5&1&1&0.5\\ 0&0.5&0.5&0.5&0 \end{bmatrix}$ iii) $\hat{\mathbf{S}} = \begin{bmatrix} 0&0&0&0&0\\ 0&0&1&1&1\\ 0&1&1&1&0\\ 0&1&1&1&0\\ 0&1&0&0&-1 \end{bmatrix}$

b)

i) $\mathbf{E} = \begin{bmatrix} 0&0&0&0&0\\ 0&1&0&0&-1\\ 0&1&0&0&-1\\ 0&1&0&0&-1\\ 0&0&0&0&0 \end{bmatrix}$ ii) $\mathbf{E} = \begin{bmatrix} 0&0&0&0&0\\ 0&1&0.5&0.5&-0.5\\ 0&0.5&0&0&-0.5\\ 0&0.5&0&0&-0.5\\ 0&-0.5&-0.5&-0.5&0 \end{bmatrix}$ iii) $\mathbf{E} = \begin{bmatrix} 0&0&0&0&0\\ 0&1&0&0&-1\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&-1&0&0&1 \end{bmatrix}$

c)

i) $$\tilde{\mathbf{S}} = \underbrace{\begin{bmatrix} 0&0&0&0&0\\ 0&0&0&0&-1\\ 0&1&0&0&-1\\ 0&1&0&0&-1\\ 0&0&0&0&0 \end{bmatrix}}_{\mathbf{E}'} + \underbrace{\begin{bmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&1&1&1\\ 0&0&1&1&1\\ 0&0&0&0&0 \end{bmatrix}}_{\hat{\mathbf{S}}'} = \begin{bmatrix} 0&0&0&0&0\\ 0&0&0&0&-1\\ 0&1&1&1&0\\ 0&1&1&1&0\\ 0&0&0&0&0 \end{bmatrix} \Rightarrow \mathbf{S}-\tilde{\mathbf{S}} = \begin{bmatrix} 0&0&0&0&0\\ 0&1&1&1&1\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{bmatrix}$$

Impulse response of the predictor filter: z^{-1}, i.e. $\hat{\mathbf{S}}$ is generated step by step from $\tilde{\mathbf{S}}$ by a shift of one sample. Feedback of the wrong prediction resulting by one transmission error – the deviation is the convolution of the transmission error by the impulse response of the inverse prediction error filter (synthesis filter), $B(z)=1/(1-z^{-1}) \Rightarrow b(n)$ is a discrete unit step function horizontally. Subsequently, the matrices for cases of both 2D predictors are given, which allow to better understand the effect of 2D error propagation.

ii)

$$\tilde{\mathbf{S}} = \underbrace{\begin{bmatrix} 0&0&0&0&0\\ 0&0&0.5&0.5&-0.5\\ 0&0.5&0&0&-0.5\\ 0&0.5&0&0&-0.5\\ 0&-0.5&-0.5&-0.5&0 \end{bmatrix}}_{\mathbf{E}'} + \underbrace{\begin{bmatrix} 0&0&0&0&0\\ 0&0&0&0.25&0.375\\ 0&0&0.5&0.625&0.25\\ 0&0.25&0.625&0.625&0.375\\ 0&0.375&0.25&0.375&-0.125 \end{bmatrix}}_{\hat{\mathbf{S}}'}$$

$$= \begin{bmatrix} 0&0&0&0&0\\ 0&0&0.5&0.75&-0.125\\ 0&0.5&0.5&0.625&-0.25\\ 0&0.75&0.625&0.625&-0.125\\ 0&-0.125&-0.25&-0.125&-0.125 \end{bmatrix} \Rightarrow \mathbf{S}-\tilde{\mathbf{S}} = \begin{bmatrix} 0&0&0&0&0\\ 0&1&0.5&0.25&0.125\\ 0&0.5&0.5&0.375&0.25\\ 0&0.25&0.375&0.375&0.125\\ 0&0.125&0.25&0.125&0.125 \end{bmatrix}$$

iii) $$\tilde{\mathbf{S}} = \begin{bmatrix} 0&0&0&0&0\\ 0&0&0&0&-1\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&-1&0&0&1 \end{bmatrix} + \begin{bmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&-1\\ 0&0&0&0&-1\\ 0&0&-1&-1&-2 \end{bmatrix} = \begin{bmatrix} 0&0&0&0&0\\ 0&0&0&0&-1\\ 0&0&0&0&-1\\ 0&0&0&0&-1\\ 0&-1&-1&-1&-1 \end{bmatrix} \Rightarrow \mathbf{S}-\tilde{\mathbf{S}} = \begin{bmatrix} 0&0&0&0&0\\ 0&1&1&1&1\\ 0&1&1&1&1\\ 0&1&1&1&1\\ 0&1&1&1&1 \end{bmatrix}$$

d)

$$\mathbf{E} = \mathbf{S}-\hat{\mathbf{S}} = \begin{bmatrix} 0&0&0&0&0\\ 0&1&2/3&1/3&-1\\ 0&1&2/3&1/3&-1\\ 0&1&2/3&1/3&-1\\ 0&0&0&0&0 \end{bmatrix} \qquad \tilde{\mathbf{S}} = \underbrace{\begin{bmatrix} 0&0&0&0&0\\ 0&1/3&1/3&1/3&-1/3\\ 0&1/3&1/3&1/3&-1/3\\ 0&1/3&1/3&1/3&-1/3\\ 0&0&0&0&0 \end{bmatrix}}_{\mathbf{E}_\mathrm{Q}} + \underbrace{\begin{bmatrix} 0&0&0&0&0\\ 0&0&1/3&2/3&1\\ 0&0&1/3&2/3&1\\ 0&0&1/3&2/3&1\\ 0&0&0&0&0 \end{bmatrix}}_{\hat{\mathbf{S}}} = \begin{bmatrix} 0&0&0&0&0\\ 0&1/3&2/3&1&2/3\\ 0&1/3&2/3&1&2/3\\ 0&1/3&2/3&1&2/3\\ 0&0&0&0&0 \end{bmatrix}$$

$$\Rightarrow \quad \mathbf{Q} = \mathbf{S}-\tilde{\mathbf{S}} = \mathbf{E}-\mathbf{E}_\mathrm{Q} = \begin{bmatrix} 0&0&0&0&0\\ 0&2/3&1/3&0&-2/3\\ 0&2/3&1/3&0&-2/3\\ 0&2/3&1/3&0&-2/3\\ 0&0&0&0&0 \end{bmatrix}$$

Interpretation: Feedback of quantization error. The values of **Q** appear shifted by one sample (convolution by impulse response of predictor filter, z^{-1}) in the prediction error signal **E**.

Problem 6.3

a)

$$\mathbf{I}_1=\begin{bmatrix}29&4&2&0\\-8&0&1&-1\\-2&2&-1&1\\1&5&0&0\end{bmatrix}\quad \mathbf{I}_2=\begin{bmatrix}59&6&2&0\\-11&0&0&-1\\-2&1&0&0\\0&2&0&0\end{bmatrix}$$

$$\mathbf{C}^{(q)}{}_1=\begin{bmatrix}232&32&16&0\\-64&0&8&-8\\-16&16&-8&8\\8&40&0&0\end{bmatrix}\quad \mathbf{C}^{(q)}{}_2=\begin{bmatrix}236&36&16&0\\-66&0&0&-16\\-16&12&0&0\\0&32&0&0\end{bmatrix}$$

b) Case 1: 0,0,0,0,1,1,0,0,0,0,0,0,EOB. Case 2: 0,0,0,0,1,2,1,1,EOB

c) Row-wise scan:
Case 1: 11+7+5+1+9+1+3+3+5+5+3+3+3+7+1+1 bit = 68 bit $\Rightarrow$ 4.25 bit/coefficient
Case 2: 13+7+5+1+9+1+1+3+5+3+1+1+1+5+1+1 bit = 58 bit $\Rightarrow$ 3.625 bit/coefficient

d)

$$\mathbf{C}-\mathbf{C}^{(q)}{}_1=\begin{bmatrix}235&35&15&3\\-67&3&5&-9\\-17&13&-7&9\\5&37&2&1\end{bmatrix}-\begin{bmatrix}232&32&16&0\\-64&0&8&-8\\-16&16&-8&8\\8&40&0&0\end{bmatrix}=\begin{bmatrix}3&3&-1&3\\-3&3&-3&-1\\-1&-3&1&1\\-3&-3&2&1\end{bmatrix}$$

$$\mathbf{C}-\mathbf{C}^{(q)}{}_2=\begin{bmatrix}235&35&15&3\\-67&3&5&-9\\-17&13&-7&9\\5&37&2&1\end{bmatrix}-\begin{bmatrix}236&36&16&0\\-66&0&0&-16\\-16&12&0&0\\0&32&0&0\end{bmatrix}=\begin{bmatrix}-1&-1&-1&3\\-1&3&5&7\\-1&1&-7&9\\5&5&2&1\end{bmatrix}$$

e)

$$\frac{1}{UV}\sum_u\sum_v\left(c_{u,v}-c^{(q)}{}_{1,u,v}\right)^2=\frac{1}{16}\left[9\cdot3^2+1\cdot2^2+6\cdot1^2\right]=5.6875\quad\Rightarrow \mathrm{PSNR}=10\lg\frac{255^2}{5.6875}=40.58\ \text{dB}$$

$$\frac{1}{UV}\sum_u\sum_v\left(c_{u,v}-c^{(q)}{}_{2,u,v}\right)^2=\frac{1}{16}\left[1\cdot9^2+2\cdot7^2+3\cdot5^2+2\cdot3^2+1\cdot2^2+7\cdot1^2\right]=17.6875\quad\Rightarrow \mathrm{PSNR}=10\lg\frac{255^2}{17.6875}=35.65\ \text{dB}$$

$$R(D)=\frac{1}{2}\log_2\frac{\sigma_v{}^2}{D}\quad\Rightarrow R(D_1)-R(D_2)=\frac{1}{2}\log_2\left[\frac{D_2}{D_1}\right]$$

With the difference by distortion, theoretically a rate difference of 0.8184 bit could be expected. The actual difference is only 0.625 bit, which can be explained by the fact that the frequency weighting is suboptimum when the distortion criterion is the unweighted squared error.

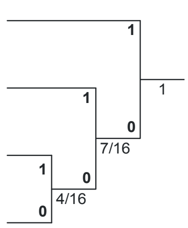
Problem 6.4

a) H=1 in case Pr(X=black)=Pr(X=white)=0.5; maximum number of bits 16·1=16.

b) $\Pr(\text{'b'})=0.25,\ \Pr(\text{'w'})=0.75\Rightarrow\ H=-\frac{1}{4}\log_2\frac{1}{4}-\frac{3}{4}\log_2\frac{3}{4}=-\log_2\frac{1}{4}-\frac{3}{4}\log_2 3\approx 2-\frac{3}{4}\cdot 1.6=0.8$

c) Pr(‘b’,’b’)=1/16, Pr(‘b’,’w’)=P(‘w’,’b’)=3/16, P(‘w’,’w’)=9/16

# bits	Symbol	Probability
1	W,W	9/16
2	W,S	3/16
3	S,W	3/16
3	S,S	1/16



$$R=\frac{1}{2}\left(\frac{9}{16}\cdot1+\frac{3}{16}\cdot2+\frac{3}{16}\cdot3+\frac{1}{16}\cdot3\right)=\frac{1}{2}\cdot\frac{27}{16}=\frac{27}{32}$$

d) i) Pr('w','w')=5/8, *P*('b','w')=*P*('w','b')=*P*('b','b')=1/8
ii) *P*('w','w')=1/2, *P*('b','w')=1/2, *P*('w','b')=*P*('b','b')=0

e) i) 5+2+3+3 bit for the code +1 bit signalization = 14 bit
ii) 4+8 bit for the code +1 bit signalization = 13 bit (better)
Advantage of ii) would e.g. not apply when the Huffman code from c) is used, as the combination ('b','w') is encoded by 3 bit.

Problem 6.5

a)
$$\sigma_v^2 = \sigma_s^2\left(1-\rho^2\right) = 16\cdot(1-\tfrac{9}{16}) = 7 \Rightarrow R(D_{\min}) = \tfrac{1}{2}\log_2\frac{\sigma_v^2}{D_{\min}} = 2 \Rightarrow \frac{\sigma_v^2}{D_{\min}} = 2^4 \Rightarrow D_{\min} = \tfrac{7}{16} \qquad G_{\text{opt}} = \frac{\sigma_s^2}{\sigma_v^2} = \frac{16}{7}$$

b)
$$\mathcal{E}\left\{c_0^2\right\} = \left(\frac{\sqrt{2}}{2}\right)^2 \mathcal{E}\left\{\left[s(n)+s(n-1)\right]^2\right\} = \left(\frac{\sqrt{2}}{2}\right)^2 \sigma_s^2\left[2+2\rho\right] = \sigma_s^2\left(1+\rho\right) = 16\cdot\tfrac{7}{4} = 28$$
$$\mathcal{E}\left\{c_1^2\right\} = \left(\frac{\sqrt{2}}{2}\right)^2 \mathcal{E}\left\{\left[s(n)-s(n-1)\right]^2\right\} = \left(\frac{\sqrt{2}}{2}\right)^2 \sigma_s^2\left[2-2\rho\right] = \sigma_s^2\left(1-\rho\right) = 16\cdot\tfrac{1}{4} = 4$$

Coding gain $G_2 = \dfrac{\frac{1}{2}\cdot(28+4)}{\sqrt{28\cdot 4}} = \dfrac{16}{\sqrt{7\cdot 16}} \Rightarrow \dfrac{G_{\text{opt}}}{G_2} = \sqrt{\dfrac{16}{7}} \Rightarrow D_{2\,\text{bit}} = D_{\min}\sqrt{\dfrac{16}{7}} = \sqrt{\dfrac{7}{16}}$

c) i) optimum: $R = \dfrac{1}{2}\log_2\dfrac{\sigma_v^2}{D} = \dfrac{1}{2}\log_2 7 \approx 1.4$
ii) using **T** :
$$R = \tfrac{1}{2}\left[\tfrac{1}{2}\log_2\frac{\mathcal{E}\{c_0^2\}}{D} + \tfrac{1}{2}\log_2\frac{\mathcal{E}\{c_1^2\}}{D}\right] = \tfrac{1}{4}\left[\log_2 28 + \log_2 4)\right] = \tfrac{1}{4}\left[\log_2 7 + 2 + 2)\right] \approx 1.7$$

d)
$$\mathcal{E}\left\{c_{00}^2\right\} = \sigma_s^2\left(1+\rho\right)^2 = 49 \quad ; \quad \mathcal{E}\left\{c_{01}^2\right\} = \mathcal{E}\left\{c_{10}^2\right\} = \sigma_s^2\left(1+\rho\right)\left(1-\rho\right) = 7$$
$$\mathcal{E}\left\{c_{11}^2\right\} = \sigma_s^2\left(1-\rho\right)^2 = 1 \Rightarrow G_{2,2} = \frac{\frac{1}{4}\left(49+7+7+1\right)}{\sqrt[4]{49\cdot 7\cdot 7\cdot 1}} = \frac{16}{7}$$

Problem 7.1

a) $(7.13) \Rightarrow \Phi_{ee}(f_1,f_2) = 2\Phi_{ss}(f_1,f_2)\left[1-\operatorname{Re}\left\{\mathcal{F}\left(p_{k_\varepsilon}(\alpha_1,\alpha_2)\right)\right\}\right]$
$$\mathcal{F}\left\{p_{k_{\varepsilon,i}}(\alpha_i)\right\} = \frac{1}{3}\left[e^{j2\pi f_i} + 1 + e^{-j2\pi f_i}\right] = \frac{1}{3}\left[1+2\cos\left(2\pi f_i\right)\right]\ ;\ i=1,2$$
$$\Rightarrow \Phi_{ee}(f_1,f_2) = 2\Phi_{ss}(f_1,f_2)\left[1-\frac{1}{9}\left(1+2\cos\left(2\pi f_1\right)\right)\left(1+2\cos\left(2\pi f_2\right)\right)\right]$$

b) Condition in the stop band of the filter: $\Phi_{ss}(f_1) < \Phi_{ss}(f_1)$.
Cutoff frequency $f_1 : \Phi_{ss}(f_1) = \Phi_{ee}(f_1)$
$$\Phi_{ss}(f_1) = \Phi_{ee}(f_1) = 2\Phi_{ss}(f_1)\left[1-\frac{1}{3}\left(1+2\cos\left(2\pi f_1\right)\right)\right] = \frac{4}{3}\Phi_{ss}(f_1)\cdot\left[1-\cos\left(2\pi f_1\right)\right]$$
$$\Rightarrow 1 = \frac{4}{3}-\frac{4}{3}\cos\left(2\pi f_1\right) \Rightarrow \frac{1}{3} = \frac{4}{3}\cos\left(2\pi f_1\right) \Rightarrow \cos\left(2\pi f_1\right) = \frac{1}{4} \Rightarrow f_1 \approx 0.21$$

c) Note: Error in problem formulation. Periodic power density spectrum should be $\phi_{ss}(f)=A|1-2f|,\ |f|\le 1/2$
$$\sigma_s^2 = 2A\int_0^{1/2}\left(1-2f_1\right)\mathrm{d}f_1 = 2A\left[\frac{1}{2}-\frac{1}{4}\right] = 0.5A$$
$$\sigma_e^2 = \frac{8A}{3}\int_0^{1/2}\left(1-2f_1\right)\cdot\left[1-\cos\left(2\pi f_1\right)\right]\mathrm{d}f_1 = \frac{8A}{3}\left[\frac{1}{4}-\int_0^{1/2}\cos\left(2\pi f_1\right)\mathrm{d}f_1 + 2\int_0^{1/2} f_1\cos\left(2\pi f_1\right)\mathrm{d}f_1\right]$$
$$= \frac{2A}{3}\left[1-\frac{2}{\pi}\left[\sin\left(2\pi f_1\right)\right]_0^{1/2} + \frac{4}{\pi}\left[\frac{1}{2\pi}\cos\left(2\pi f_1\right) + f_1\sin\left(2\pi f_1\right)\right]_0^{1/2}\right] = \frac{2A}{3}\left[1-\frac{4}{\pi^2}\right] \approx 0.396A$$
$$\Rightarrow G = \frac{\sigma_s^2}{\sigma_e^2} \approx 1.26\ ;\ G_{\text{SNR}} = 10\lg G \approx 1.01\ \text{dB}$$

$$\sigma_e^2 = \frac{8A}{3}\int_0^{0.21}(1-2f_1)\cdot\left[1-\cos(2\pi f_1)\right]\mathrm{d}\,f_1 + 2A\int_{0.21}^{1/2}(1-2f_1)\,\mathrm{d}\,f_1$$
$$= \frac{8A}{3}\left[0.21-0.21^2-\int_0^{0.21}\cos(2\pi f_1)\,\mathrm{d}\,f_1 + 2\int_0^{0.21} f_1\cos(2\pi f_1)\,\mathrm{d}\,f_1\right] + 2A\left[\frac{1}{2}-\frac{1}{4}-0.21+0.21^2\right]$$
$$= \frac{2A}{3}\left[0.6636 - \underbrace{\frac{2}{\pi}\left[\sin(2\pi f_1)\right]_0^{0.21}}_{0.6166} + \underbrace{\frac{4}{\pi}\left[\frac{1}{2\pi}\cos(2\pi f_1) + f_1\sin(2\pi f_1)\right]_0^{0.21}}_{0.1067}\right] + 0.1682A$$
$$= 0.1025A + 0.1682A \approx 0.2707A$$
$$\Rightarrow G = \frac{\sigma_s^2}{\sigma_{e,\mathrm{filt}}^2} \approx 1.85\ ;\ G_{\mathrm{SNR}} = 10\cdot\log_{10} G \approx 2.668\ \ \mathrm{dB}$$

Problem 7.2

a) Vector $\mathbf{k}_1$: $10^2+10^2+0^2+(-10)^2 = 300$

Vector $\mathbf{k}_2$: $20^2+20^2+20^2+20^2 = 1600$

Vector $\mathbf{k}_1$ is the better choice according to this criterion.

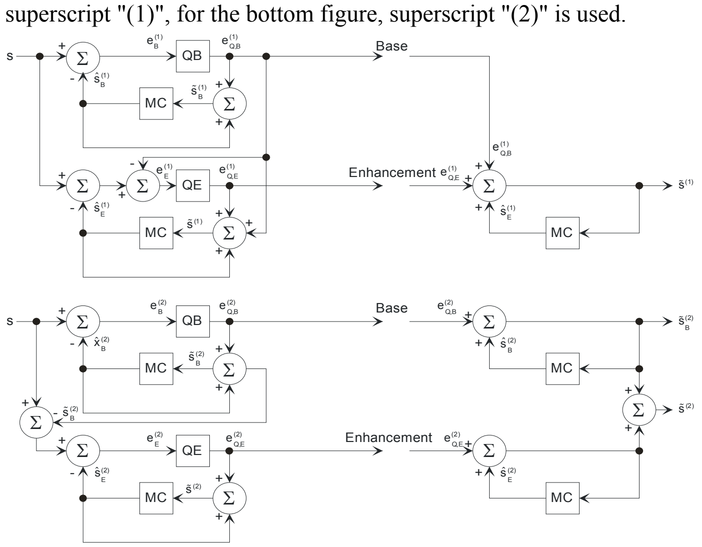
b)
$$\begin{bmatrix}\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\end{bmatrix}\cdot\begin{bmatrix}10 & 10\\ 0 & -10\end{bmatrix}\cdot\begin{bmatrix}\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\end{bmatrix} = \begin{bmatrix}5\cdot\sqrt{2} & 0\\ 5\cdot\sqrt{2} & 10\cdot\sqrt{2}\end{bmatrix}\cdot\begin{bmatrix}\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\end{bmatrix} = \begin{bmatrix}5 & 5\\ 15 & -5\end{bmatrix}$$
$$\begin{bmatrix}\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\end{bmatrix}\cdot\begin{bmatrix}20 & 20\\ 20 & 20\end{bmatrix}\cdot\begin{bmatrix}\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\end{bmatrix} = \begin{bmatrix}20\cdot\sqrt{2} & 20\cdot\sqrt{2}\\ 0 & 0\end{bmatrix}\cdot\begin{bmatrix}\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\end{bmatrix} = \begin{bmatrix}40 & 0\\ 0 & 0\end{bmatrix}$$

Bit allocation in case of $\mathbf{k}_1$: 7 + 7 + 9 + 7 bit = 30 bit
Bit allocation in case of $\mathbf{k}_2$: 13 + 1 + 1 + 1 bit = 13 bit

In both cases lossless coding of the (discrete) prediction error signal. Choosing $\mathbf{k}_2$ gives a much lower number of bits, which can be explained by the high spatial correlation of the prediction error signal.

Problem 7.3

In the following block diagram the notation definitions used for the signals are given. For the top figure, the signals are marked by superscript "(1)", for the bottom figure, superscript "(2)" is used.



All the subsequent formulations are made in the spectral domain, where the motion-compensated predictor is defined as a linear system of transfer function $H_{\mathrm{MC}}(\mathbf{f})$. The base layer is identical for both structures. Hence, the following relationships hold:

$$\hat{S}_{\mathrm{B}}^{(1)}(\mathbf{f}) = \hat{S}_{\mathrm{B}}^{(2)}(\mathbf{f}) = \hat{S}_{\mathrm{B}}(\mathbf{f})\quad;\quad E_{\mathrm{B}}^{(1)}(\mathbf{f}) = E_{\mathrm{B}}^{(2)}(\mathbf{f}) = E_{\mathrm{B}}(\mathbf{f})$$
$$E_{\mathrm{Q,B}}^{(1)}(\mathbf{f}) = E_{\mathrm{Q,B}}^{(2)}(\mathbf{f}) = E_{\mathrm{Q,B}}(\mathbf{f})\quad;\quad \tilde{S}_{\mathrm{B}}^{(1)}(\mathbf{f}) = \tilde{S}_{\mathrm{B}}^{(2)}(\mathbf{f}) = \tilde{S}_{\mathrm{B}}(\mathbf{f})$$

Further, as both reconstructed signals shall be equal:

$$\tilde{S}(\mathbf{f}) = \tilde{S}^{(1)}(\mathbf{f}) = \left[E_{\mathrm{Q,B}}(\mathbf{f}) + E_{\mathrm{Q,E}}^{(1)}(\mathbf{f})\right]\frac{1}{1-H_{\mathrm{MC}}(\mathbf{f})} = \tilde{S}^{(2)}(\mathbf{f}) = \frac{E_{\mathrm{Q,B}}(\mathbf{f})}{1-H_{\mathrm{MC}}(\mathbf{f})} + \frac{E_{\mathrm{Q,E}}^{(2)}(\mathbf{f})}{1-H_{\mathrm{MC}}(\mathbf{f})}$$

$$\Rightarrow E_{\mathrm{Q,E}}^{(1)}(\mathbf{f}) = E_{\mathrm{Q,E}}^{(2)}(\mathbf{f}) = E_{\mathrm{Q,E}}(\mathbf{f})$$

This leads to the following definitions of prediction estimates:

$$\hat{S}_{\mathrm{E}}^{(1)}(\mathbf{f}) = H_{\mathrm{MC}}(\mathbf{f})\frac{E_{\mathrm{Q,B}}(\mathbf{f}) + E_{\mathrm{Q,E}}(\mathbf{f})}{1-H_{\mathrm{MC}}(\mathbf{f})} \quad \text{and} \quad \hat{S}_{\mathrm{E}}^{(2)}(\mathbf{f}) = H_{\mathrm{MC}}(\mathbf{f})\frac{E_{\mathrm{Q,E}}(\mathbf{f})}{1-H_{\mathrm{MC}}(\mathbf{f})}$$

$$\Rightarrow \hat{S}_{\mathrm{E}}^{(1)}(\mathbf{f}) - \hat{S}_{\mathrm{E}}^{(2)}(\mathbf{f}) = H_{\mathrm{MC}}(\mathbf{f})\frac{E_{\mathrm{Q,B}}(\mathbf{f})}{1-H_{\mathrm{MC}}(\mathbf{f})} = H_{\mathrm{MC}}(\mathbf{f})\cdot\tilde{S}_{\mathrm{B}}(\mathbf{f})$$

Finally, the prediction errors at the enhancement quantizer inputs are:

$$\begin{aligned}
E_{\mathrm{E}}^{(1)}(\mathbf{f}) &= S(\mathbf{f}) - \hat{S}_{\mathrm{E}}^{(1)}(\mathbf{f}) - E_{\mathrm{Q,B}}(\mathbf{f}) \\
&= S(\mathbf{f}) - H_{\mathrm{MC}}(\mathbf{f})\frac{E_{\mathrm{Q,B}}(\mathbf{f}) + E_{\mathrm{Q,E}}(\mathbf{f})}{1-H_{\mathrm{MC}}(\mathbf{f})} - E_{\mathrm{Q,B}}(\mathbf{f}) \\
&= S(\mathbf{f}) - H_{\mathrm{MC}}(\mathbf{f})\cdot\frac{E_{\mathrm{Q,E}}(\mathbf{f})}{1-H_{MC}(\mathbf{f})} - E_{\mathrm{Q,B}}(\mathbf{f})\underbrace{\left[1 + \frac{H_{\mathrm{MC}}(\mathbf{f})}{1-H_{\mathrm{MC}}(\mathbf{f})}\right]}_{=\frac{1}{1-H_{MC}(\mathbf{f})}} \\
&= S(\mathbf{f}) - H_{\mathrm{MC}}(\mathbf{f})\frac{E_{\mathrm{Q,E}}(\mathbf{f})}{1-H_{\mathrm{MC}}(\mathbf{f})} - \tilde{S}_{\mathrm{B}}(\mathbf{f})
\end{aligned}$$

$$\begin{aligned}
E_{\mathrm{E}}^{(2)}(\mathbf{f}) &= S(\mathbf{f}) - \hat{S}_{\mathrm{E}}^{(2)}(\mathbf{f}) - \tilde{S}_{\mathrm{B}}(\mathbf{f}) \\
&= S(\mathbf{f}) - H_{\mathrm{MC}}(\mathbf{f})\frac{E_{\mathrm{Q,E}}(\mathbf{f})}{1-H_{\mathrm{MC}}(\mathbf{f})} - \tilde{S}_{\mathrm{B}}(\mathbf{f}) \\
&\Rightarrow E_{\mathrm{E}}^{(1)}(\mathbf{f}) = E_{\mathrm{E}}^{(2)}(\mathbf{f}) = E_{\mathrm{E}}(\mathbf{f})
\end{aligned}$$

Consequently, not necessary to implement two recursive loops in the decoder, if only the enhancement layer signal shall be decoded.

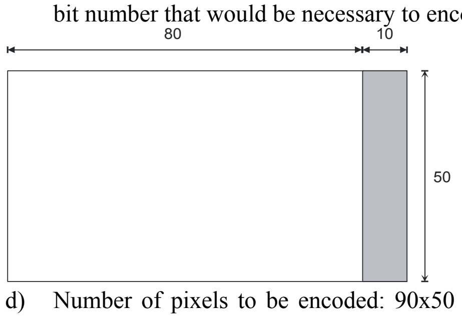
Problem 7.4

a) $\frac{1}{2}\log_2\frac{\sigma_s^2}{D} = 6 \Rightarrow D = \frac{\sigma_s^2}{2^{12}} = \frac{1}{256}$

b) $R_{\mathrm{G,1D}} = -\frac{1}{2}\log_2(1-\rho^2) = \frac{1}{2}\log_2(4) = 1\text{ bit}$

$R_{\mathrm{G,2D}} = 2R_{\mathrm{G,1D}} = 2\text{ bit}$

c) The shaded area is uncovered and has to be newly encoded. This is 1/9 of the total area, and would on average require 1/9 of the bit number that would be necessary to encode a complete frame by intraframe coding.



d) Number of pixels to be encoded: 90x50 (1st frame) + 90x50x1/9 x9 (subsequent frames) =9000; 9000 pixel x 4 bit/pixel = 36000 bit.

e) 4500 pixel in 1st frame ⇒ 9000/4500=2 bit/pixel. Considering the coding gain by utilizing the spatial correlation:

$$D_1=\frac{\sigma_s{}^2}{2^{2R_1}}\cdot(1-\rho_1{}^2)(1-\rho_2{}^2)=\frac{\sigma_s{}^2}{2^4}\cdot\frac{1}{16}=\frac{1}{16}\Rightarrow\frac{D_1}{D}=\frac{\frac{1}{16}}{\frac{1}{256}}=16$$

f) Coding distortion from the previous frame is fed into the prediction error signal. The spectrum of the prediction error signal then is, according to (7.14) $\Phi_{ee}(f_1,f_2)=\min(\Phi_{ss}(f_1,f_2),\Theta)$

The coding distortion from e) is white noise, as according to the following estimation condition (11.23) still holds, i.e. encoding is performed within the range of "high distortion":

$$D_1\overset{?}{\leq}\frac{(1-\rho_1)(1-\rho_2)}{(1+\rho_1)(1+\rho_2)}\cdot 16=\frac{\left(1-\sqrt{3}/2\right)^2}{\left(1+\sqrt{3}/2\right)^2}\cdot 16=\frac{7/4-\sqrt{3}}{7/4+\sqrt{3}}\cdot 16$$

$$=\left(7/4-\sqrt{3}\right)^2\cdot 256\approx 0.0004\cdot 256\approx 0.1>\frac{1}{16}$$

Hence in this case, $\Theta=D_1$ and $\Phi_{ee}(f_1,f_2)=D_1=$const.

Problem 7.5

For simplification, a one-dimensional case and compensation errors k_ε are used subsequently; extension to multiple dimensions is straightforward. From (7.19),

$$\begin{aligned}\left|E_{\mathrm{b}}(f)\right|^2&=\left|S(f)\right|^2\left[\mathrm{j}e^{-\mathrm{j}\pi fk_{\varepsilon,-1}}\sin\left(\pi fk_{\varepsilon,-1}\right)+\mathrm{j}e^{-\mathrm{j}\pi fk_{\varepsilon,+1}}\sin\left(\pi fk_{\varepsilon,+1}\right)\right]\left[-\mathrm{j}e^{\mathrm{j}\pi fk_{\varepsilon,-1}}\sin\left(\pi fk_{\varepsilon,-1}\right)-\mathrm{j}e^{\mathrm{j}\pi fk_{\varepsilon,+1}}\sin\left(\pi fk_{\varepsilon,+1}\right)\right]+\left|R(f)\right|^2\\&=\left|S(f)\right|^2\left[\sin^2\left(\pi fk_{\varepsilon,-1}\right)+\sin^2\left(\pi fk_{\varepsilon,+1}\right)+2\sin\left(\pi fk_{\varepsilon,-1}\right)\sin\left(\pi fk_{\varepsilon,+1}\right)\left[\cos\left(\pi fk_{\varepsilon,-1}\right)\cos\left(\pi fk_{\varepsilon,+1}\right)+\sin\left(\pi fk_{\varepsilon,-1}\right)\sin\left(\pi fk_{\varepsilon,+1}\right)\right]\right]+\end{aligned}$$

From this, in case of statistically independent and zero-mean deviations $k_{\varepsilon,+1}$ and $k_{\varepsilon,-1}$, the cross-products come to zero when the expectation is computed. Therefore,

$$\begin{aligned}\phi_{e_\mathrm{b}e_\mathrm{b}}(f)&=\phi_{ss}(f)\mathcal{E}\left\{\left[\sin^2\left(\pi fk_{\varepsilon,-1}\right)+\sin^2\left(\pi fk_{\varepsilon,+1}\right)\right]\right\}+\phi_{rr}(f)=\phi_{ss}(f)\left[\frac{1}{2}-\frac{1}{2}\mathcal{E}\left\{\cos\left(2\pi fk_{\varepsilon,+1}\right)\right\}+\frac{1}{2}-\frac{1}{2}\mathcal{E}\left\{\cos\left(2\pi fk_{\varepsilon,-1}\right)\right\}\right]\\&=\phi_{ss}(f)\left[1-\frac{1}{2}\int_{-\infty}^{\infty}p_{k_{\varepsilon,+1}}(\alpha)\cos\left(2\pi f\alpha\right)\mathrm{d}\alpha-\frac{1}{2}\int_{-\infty}^{\infty}p_{k_{\varepsilon,-1}}(\alpha)\cos\left(2\pi f\alpha\right)\mathrm{d}\alpha\right]+\phi_{rr}(f)\\&=\phi_{ss}(f)\left[1-\mathrm{Re}\left\{\boldsymbol{\mathcal{F}}\left\{p_{k_\varepsilon}(\alpha)\right\}\right\}\right]+\phi_{rr}(f)\text{ when }p_{k_{\varepsilon,+1}}(\alpha)=p_{k_{\varepsilon,-1}}(\alpha).\end{aligned}$$

When $k_{\varepsilon,+1}=k_{\varepsilon,-1}=k_\varepsilon$, the result becomes identical to uni-directional prediction in (7.9), which is explainable by the fact that systematically the same wrong compensation is performed from both preceding and subsequent frames.

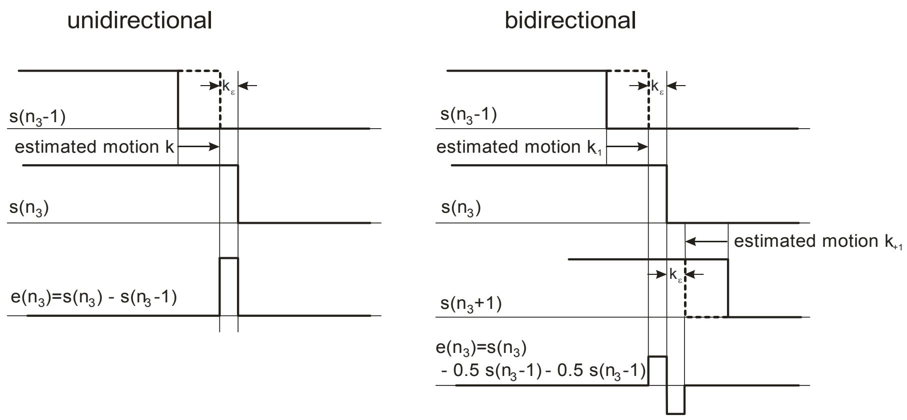
$$\begin{aligned}\phi_{e_\mathrm{b}e_\mathrm{b}}(f)&=\phi_{ss}(f)\mathcal{E}\left\{\left[2\sin^2\left(\pi fk_\varepsilon\right)+2\sin^2\left(\pi fk_\varepsilon\right)\left[\cos^2\left(\pi fk_\varepsilon\right)+\sin^2\left(\pi fk_\varepsilon\right)\right]\right]\right\}+\phi_{rr}(f)\\&=2\phi_{ss}(f)\left[1-\mathcal{E}\left\{\cos\left(2\pi fk_\varepsilon\right)\right\}\right]+\phi_{rr}(f)=2\phi_{ss}(f)\left[1-\mathrm{Re}\left\{\boldsymbol{\mathcal{F}}\left\{p_{k_\varepsilon}(\alpha)\right\}\right\}\right]+\phi_{rr}(f)\ .\end{aligned}$$

When $k_{\varepsilon,+1}=-k_{\varepsilon,-1}=k_\varepsilon$,

$$\begin{aligned}\phi_{e_\mathrm{b}e_\mathrm{b}}(f)&=\phi_{ss}(f)\mathcal{E}\left\{\left[2\sin^2\left(\pi fk_\varepsilon\right)-2\sin^2\left(\pi fk_\varepsilon\right)\left[\cos^2\left(\pi fk_\varepsilon\right)-\sin^2\left(\pi fk_\varepsilon\right)\right]\right]\right\}+\phi_{rr}(f)\\&=\phi_{ss}(f)\mathcal{E}\left\{\left[2\sin^2\left(\pi fk_\varepsilon\right)-2\sin^2\left(\pi fk_\varepsilon\right)\left[1-2\sin^2\left(\pi fk_\varepsilon\right)\right]\right]\right\}+\phi_{rr}(f)\\&=\phi_{ss}(f)\mathcal{E}\left\{\left[4\sin^4\left(\pi fk_\varepsilon\right)\right]\right\}+\phi_{rr}(f)=2\phi_{ss}(f)\mathcal{E}\underbrace{\left\{\left[1-\cos^2\left(2\pi fk_\varepsilon\right)\right]\right\}}_{=\sin^2\left(2\pi fk_\varepsilon\right)}+\phi_{rr}(f)\\&=\phi_{ss}(f)\left[1-\mathcal{E}\left\{\cos\left(4\pi fk_\varepsilon\right)\right\}\right]+\phi_{rr}(f)=\phi_{ss}(f)\left[1-\mathrm{Re}\left\{\boldsymbol{\mathcal{F}}\left\{p_{k_\varepsilon}(2\alpha)\right\}\right\}\right]+\phi_{rr}(f)\ .\end{aligned}$$

The PDF is becoming narrower and therefore, its Fourier transform is becoming wider. As a consequence, the increase of spectral power towards higher frequencies is typically even less than in the statistically independent case. As an interpretation, the systematic behaviour of misaligning in one direction for the preceding and in the other direction for the subsequent frame is equivalent to additional spatial lowpass filtering.

The subsequent picture illustrates the advantage of bidirectional prediction for the case of wrong motion compensation of an edge. In case of unidirectional prediction, a peak of width k_ε appears in the prediction error signal $e(n_3)$. In case of bidirectional prediction, though two peaks appear at distinct positions, their amplitude is reduced by half, such that the prediction error energy is also reduced by half.


unidirectional
bidirectional
s(n3-1)
estimated motion k
s(n3)
e(n3)=s(n3) - s(n3-1)
s(n3-1)
estimated motion k1
s(n3)
estimated motion k+1
s(n3+1)
e(n3)=s(n3)
- 0.5 s(n3-1) - 0.5 s(n3-1)